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The Light-Cone Gauge and the Principal Value Prescription—Ward Identities in Yang–Mills Theories

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Abstract. By explicit calculation of radiative corrections to the self-energy and the three-vertex at one-loop level for Yang–Mills theories in the light-cone gauge, it is demonstrated that analytically regulated Feynman integrals defined by the principal value prescription satisfy the one, two, and three-point Ward identities. However, both the calculated self-energy and three-vertex have anomalous, unrenormalizable infinite parts, thus confirming the belief, based on previous calculations of only the self-energy, that the principal value prescription is seriously flawed in the light-cone gauge.

Recently [1] it has been shown that Feynman integrals in the light-cone gauge [2], defined by the constraint* $n \cdot A_a = 0$, where A_μ^a are the gauge fields and n_μ is an arbitrary vector satisfying $n^2 = 0$ can be regulated by analytic continuation. The result for a class of Feynman integrals needed to compute general two-point functions

$$L_{2\omega}(p,n;\kappa,\mu,\bar{\nu}) \equiv \int \underline{d}^{2\omega} q [(p-q)^2]^\kappa (q^2)^\mu (q \cdot n)^{\bar{\nu}}|_{n^2=0} = \frac{\pi^2 (p^2)^{\omega+\kappa+\mu} (p \cdot n)^\nu \Gamma(\omega+\kappa) \Gamma(\omega+\mu+\bar{\nu}) \Gamma(-\omega-\kappa-\mu)}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(2\omega+\kappa+\mu+\bar{\nu})} \tag{1}$$

is incomparably simpler than that for general axial gauges ($n^2 \neq 0$) [3]. With this result** the light-cone gauge appears to display the virtue it has long been thought to possess: being an axial gauge it is ghost-free, yet its Feynman integrals are as simple as those of the ghost-infested covariant gauges. A short derivation of (1) is given in Appendix *D*. It was also shown in [1] that this result is identical to that for corresponding Feynman integrals defined by the principal value prescription.

In our method of analytic regularization, divergent Feynman integrals are defined, by analytic continuation, as limiting cases of integrals where the number of dimensions *and* exponents of quantities such as q^2, $(p-q)^2$ and $q \cdot n$ are continuous variables. The reason why this method is more powerful than the widely used method of dimensional regularization [4], where only the number of dimensions is made continuous, is explained in [1]. However, there appears to be a widespread point of view [5] that gauge invariance is not preserved in a general analytic regularization, except in dimensional regularization. The purpose of this work is to demonstrate, by explicit verification of Ward identities, that contrary to the belief just mentioned, our method does preserve the gauge invariance of Yang–Mills theories. We succeed in achieving this goal.

The Yang–Mills self-energy in the light-cone gauge has already been calculated with the principal value prescription by several groups [6, 7] and has been shown to contain anomalous infinite parts of $O(g^2)$. Our calculation, based on an entirely different method, confirms the earlier results. The three-point vertex is calculated here for the first time and is shown to have anomalous infinite parts of $O(g^3)$, thus reinforcing the belief that the principal value prescription does not give a renormalizable Yang–Mills theory in the light-cone gauge.

The suggestion that analytic regularization is associated with the violation of gauge invariance possibly arises from the work of Speer [8]. In Speer's method quantum field-theory is regulated by modifying propagators: replacing, say, $[(p-q)^2]^{-1}$ by

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* We work in the Euclidean space throughout, with metric $g_{\mu\nu} = \delta_{\mu\nu} = (1,1,1,1)$. Minkowski space is reached by analytic continuation. Early Latin superscripts are indices of the gauge group and middle Greek subscripts are Lorentz indices. The structure constants of the gauge group are defined in terms of the commutator of the generators $[t^a, t^b] = f^{abc} t^c$. The scalar product acts in the Lorentz space $n \cdot A^a \equiv n_\mu A_\mu^a$; and the tensor product acts in the gauge-group space $(A_\mu \wedge A_\nu)^a \equiv f^{abc} A_\mu^b A_\nu^c$. Exponents in the representation (1) are not to be confused with Lorentz indices

** Note the right-hand-side of (1) is *defined* to be proportional to π^2, instead of the usual π^ω. This gives a simpler result in the $\varepsilon = \omega - 2 \to 0$ limit—$\ln \pi$ terms are absent. For κ, μ and ν being integers, the limiting process we use to evaluate (1) is to let $(\kappa, \mu, \bar{\nu})$ = integers first, then let ε approach zero

$[(p-q)^2]^\lambda$. This prescription, much like that of Pauli and Villars [9], defines a new theory which is ultraviolet-finite. The real theory is recovered in the limit $\lambda \to -1$. However, since the structure of a Lagrangian with a propagator having a continuous exponent is not known, gauge transformation is not a well defined concept in Speer's modified theory; in this sense gauge invariance is violated. On the surface Speer's modification of the propagator appears to be identical to the continuation of exponents in our method. There is, however, a subtle yet crucial difference in the two approaches. In our method, infinities of the theory are controlled by regulating, by analytic continuation, divergent Feynman *integrals*, which are the only places where infinities occur; the propagator of, say, a massless vector boson is still $(p-q)^{-2}$, not $(p-q)^{2\lambda}$. This being the case, and since our method obeys the basic rules of algebra such as associativity and commutativity of operators, Ward identities can be formally derived as usual, and are expected to be satisfied.

To demonstrate this explicitly consider the generating functional [10] for a non-Abelian Yang–Mills theory

$$Z[\eta] = \int [dA] \exp iS[\eta] \tag{2}$$

with the action (see footnote* on first page)

$$S[\eta] = \int d^4x (\mathscr{L}_{\rm YM} + \mathscr{L}_\xi + \eta^a \cdot A^a) \tag{3}$$

where η^a_μ is the source field,

$$\begin{aligned} \mathscr{L}_{\rm YM} &= -\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g(A_\mu \times A_\nu)^a \end{aligned} \tag{4}$$

is the Yang–Mills Lagrangian and

$$\mathscr{L}_\xi = -\frac{1}{2\xi}(n\cdot A^a)^2 \tag{5}$$

is the gauge-fixing term [11]. From (2), (3) and (5) it is seen that the axial gauge condition $n\cdot A^a = 0$ is realized in the limit $\xi \to 0$. In (3–5), as will be the practice elsewhere in this article, the space-time dependence of the fields η^a_μ and A^a_μ has been suppressed. The term quadratic in A^a_μ in the total Lagrangian $\mathscr{L} = \mathscr{L}_{\rm YM} + \mathscr{L}_\xi$ gives the $O(g^0)$ self-energy in momentum space

$$\begin{aligned} \Pi^{(0)ab}_{\mu\nu}(p) &\equiv \delta^{ab} \Pi^{(0)}_{\mu\nu}(p) \\ &= -i\delta^{ab}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu - \frac{1}{\xi} n_\mu n_\nu\right) \end{aligned} \tag{6}$$

The ξ-dependent term coming from $\mathscr{L}_\xi$ is important: without it the free propagator $\Delta^{(0)}$, which is the reciprocal of $\Pi^{(0)}$, does not exist. As (6) stands,

$$\begin{aligned} &\Delta^{(0)ab}_{\mu\nu}(p) \\ &\equiv \delta^{ab}\Delta^{(0)}_{\mu\nu}(p) \equiv \int d^4x e^{-ip(x-y)} \langle A^a_\mu(x) A^b_\nu(y)\rangle|_{g=0} \\ &= i\frac{\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + (n^2 - \xi p^2)\frac{p_\mu p_\nu}{(p\cdot n)^2}\right] \end{aligned} \tag{7}$$

Then an amputated n-point function ${}_n\Gamma(p_1, \ldots, p_n)$, is given by

$$\begin{aligned} &(2\pi)^4 \delta^4(p_1 + \cdots + p_n)\left[\prod_{i=1}^n \Delta^{(0)}_{\alpha_i\beta_i}(p_i)\right] \\ &\cdot {}_n\Gamma_{\beta_1\ldots\beta_n}(p_1,\ldots,p_n) = \left[\prod_{i=1}^n \int d^4x_i e^{-ip_i x_i}\right] \\ &\cdot \langle A_{\alpha_1}(x_1)\ldots A_{\alpha_n}(x_n)\rangle \end{aligned} \tag{8}$$

where each of the early Greek indices $\alpha_i, \beta_i, \ldots$ stand for a Lorentz index and a gauge-group index. The symbol $\Pi_{\alpha\beta}$ will be reserved for the amputed two-point function, i.e. the proper self-energy. From (8), the lowest order three- and four-point functions are

$$\begin{aligned} &{}_3\Gamma^{(0)abc}_{\lambda\mu\nu}(p,q,r) \\ &= g f^{abc}[\delta_{\lambda\mu}(p-q)_\nu + \delta_{\mu\nu}(q-r)_\lambda + \delta_{\nu\lambda}(r-p)_\mu] \end{aligned} \tag{9}$$

$$\begin{aligned} {}_4\Gamma^{(0)abcd}_{\mu\nu\rho\sigma}(p,q,r,s) = &-ig^2[f^{abe}f^{cde}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \\ &+ f^{ace}f^{bde}(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\rho\nu}) \\ &+ f^{ade}f^{cbe}(\delta_{\mu\rho}\delta_{\sigma\nu} - \sigma_{\mu\nu}\delta_{\sigma\rho})] \end{aligned} \tag{10}$$

These are identical to the corresponding functions in covariant gauges.

The free propagator $\Delta^{(0)}$ is transverse to n_μ in the limit $\xi \to 0$:

$$n_\mu \Delta^{(0)}_{\mu\nu}(p) = \Delta^{(0)}_{\nu\mu} n_\mu = -i\xi p\sqrt{(p\cdot n)} \tag{11}$$

From this and (8), any contraction of an unamputated n-point function $\langle A_{\alpha_1} \ldots A_{\alpha_n}\rangle$ with n_μ vanishes in the limit $\xi \to 0$:

$$\lim_{\xi\to 0} \langle A_{\alpha_1} \ldots (n\cdot A) \ldots A_{\alpha_n}\rangle = 0 \tag{12}$$

This means that the axial gauge condition, $n\cdot A^a = 0$, is realized in the limit $\xi \to 0$. The $O(1/\xi)$ term in $\Pi^{(0)}_{\mu\nu}$ of (6) may cause uneasiness in this limit. However, since this term is unrenormalized, it is decoupled from everything else and therefore does not cause any real trouble. The easiest way to understand this is to recognize that the ξ-dependence of all corrections to $\Pi^{(0)}$ comes solely from $\Delta^{(0)}$, so that the sum of such corrections must be a polynomial in ξ without negative powers. The $O(1/\xi)$ term is thus always unaffected. In this way the gauge-breaking $\mathscr{L}_\xi$ serves its intended purpose: it allows the free propagator to be constructed, after which it may be expediently set to zero. Since our purpose is to understand the structure of the theory in the light-cone gauge, we do not set ξ equal to zero at this early stage. In all our calculations ξ is in fact kept finite until the final stage.

It will be shown later that the part of $\Pi_{\mu\nu}$ that is regular in ξ is transverse to p_μ. Then $\Pi_{\mu\nu}$ may always be expressed as

$$\Pi_{\mu\nu}(p) = \Pi^t_{\mu\nu}(p) + \frac{i}{\xi} n_\mu n_\nu \tag{13a}$$

$$\Pi^t_{\mu\nu}(p) = -i(\Pi_0 P_{\mu\nu} + \Pi_1 N_{\mu\nu}) \tag{13b}$$