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L'ÉNERGIE ATOMIQUE
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YANG-MILLS THEORIES IN AXIAL AND LIGHT-CONE GAUGES, ANALYTIC REGULARIZATION AND WARD IDENTITIES

Théories de Yang-Mills pour les jauges axiales et à cône lumineux, régularisation analytique et identités de Ward

H.C. LEE

Lectures given at the Physics Centre, National Taiwan University, Taipei, 1984 March and April

Chalk River Nuclear Laboratories

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par

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Résumé

On commente dans ce rapport l'application des principes de généralisation et de continuation analytique à la régularisation des intégrales divergentes de Feynman. Cette technique, appelée régularisation analytique, est une généralisation de la régularisation dimensionnelle. Elle permet d'effectuer des représentations analytiques pour deux catégories d'intégrales à deux points et sans masse. La première catégorie est fondée sur la prescription d'une valeur principale et elle comprend des intégrales mesurées dans les théories des champs quantiques au moyen de jauges axiales sans signal fantôme ($n \cdot A = 0$), intégrales pouvant être transformées exceptionnellement en intégrales mesurables au moyen de jauges à cône lumineux ($n \cdot A = 0$, $n^2 = 0$). La deuxième catégorie est fondée sur la prescription de Mandelstam conçue spécialement pour les jauges à cône lumineux. Pour certaines intégrales mesurées au moyen de jauges à cône lumineux, les deux représentations ne sont pas équivalentes. Les deux catégories comportent des intégrales sous-catégorielles mesurées par des "jauges ξ " covariantes de Lorentz. Les représentations permettent: de calculer les corrections d'une seule boucle devant être apportées à l'énergie propre et aux trois sommets, selon les théories de Yang-Mills, pour les jauges axiales et à cône lumineux, pour répondre aux exigences des identités Ward à deux et à trois points; d'illustrer le fait que les particularités ultraviolettes et infrarouges, indiscernables dans la régularisation dimensionnelle, peuvent être séparées analytiquement; et de montrer que certaines intégrales Tadpole disparaissent parce que les particularités ultraviolettes et infrarouges s'annulent complètement. Dans la jauge axiale, les constantes de renormalisation des sommets et des fonctions d'onde, à savoir Z_3 et Z_1, sont identiques de sorte que la fonction β peut provenir directement de Z_3 (c'est-à-dire de l'énergie propre) le résultat étant le même que celui obtenu dans les jauges ξ covariantes. Les résultats préliminaires semblent indiquer que les jauges à cône lumineux employées dans le cas de la prescription Mandelstam et non dans le cas de la prescription à valeur principale, ont la même propriété de renormalisation que les jauges axiales.

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H.C. Lee

A b s t r a c t

The application of the principles of generalization and analytic continuation to the regularization of divergent Feynman integrals is discussed. The technique, or analytic regularization, which is a generalization of dimensional regularization, is used to derive analytic representations for two classes of massless two-point integrals. The first class is based on the principal-value prescription and includes integrals encountered in quantum field theories in the ghost-free axial gauge ($n \cdot \underline{A} = 0$), reducing in a special case to integrals in the light-cone gauge ($n \cdot \underline{A} = 0$, $n^2 = 0$). The second class is based on the Mandelstam prescription devised especially for the light-cone gauge. For some light-cone gauge integrals the two representations are not equivalent. Both classes include as a subclass integrals in the Lorentz covariant "ξ-gauges". The representations are used to compute one-loop corrections to the self-energy and the three-vertex in Yang-Mills theories in the axial and light-cone gauges, showing that the two- and three-point Ward identities are satisfied; to illustrate that ultraviolet and infrared singularities, indistinguishable in dimensional regularization, can be separated analytically; and to show that certain tadpole integrals vanish because of an exact cancellation between ultraviolet and infrared singularities. In the axial gauge, the wavefunction and vertex renormalization constants, Z_3 and Z_1, are identical, so that the β-function can be directly derived from Z_3 (i.e. from the self-energy), the result being the same as that computed in the covariant ξ-gauges. Preliminary results suggest that the light-cone gauge in the Mandelstam prescription, but not in the principal value prescription, has the same renormalization property of the axial gauge.

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PREFACE

These notes are based on lectures given at the Physics Centre, National Taiwan University, in March and April, 1984, and at a number of other places in China and Japan in May.

The lectures deal at a pedagogical level with topics related to the need for, and methods employed to the regularization of divergent Feynman integrals in quantum field theories. The major part of these notes is devoted to the development and application of a new analytic regularization technique.

Although regularization involves relatively simple mathematical concepts and techniques, it is not unusual that a student does not learn about renormalization - perhaps the single most important topic setting quantum field theory apart from its classical counterpart - because he is intimidated by divergent integrals he encounters but cannot deal with. With the advent of the method of dimensional regularization, the evaluation of divergent Feynman integral has for the most part become routine. In these lectures we discuss a recently developed generalization of this method that we call analytic regularization. In essence it is a hybrid of an older method bearing the same name and the dimensional method. In developing the analytic method the two-step procedure of generalization and analytic continuation is given special emphasis. Among the advantages of taking such a systematic approach is the reward of finding representations for classes of Feynman integrals that are extremely easy to evaluate. The power of this approach is especially manifest in dealing with integrals of Yang-Mills theories in the ghost-free axial and light-cone gauges.

An often sought after property of a regularization method is the preservation of symmetries in the associated field theory, some examples of which are gauge invariance, the Becchi-Rouet-Stora invariance and supersymmetries. With this goal in mind, these notes follow a program whereby the operations of tensor algebra and the regularization of integrals are separated as much as possible. In such a program only formally invariant, or scalar, integrals need be regularized. From the point of view of such an approach, the dimensional method is a purely formal technique which need not be associated with "doing physics in 2ω dimensions". Thus the trace of the Euclidean metric is equal to the dimension d (an integer), not to the generalized dimension 2ω (a continuous variable). This approach is the generalization of the one known in the literature as dimension reduction. Although we believe it shows great promise, in these notes we have only shown that it preserves guage invariance, at least at the one-loop level.

The reader is assumed to have a rudimentary knowledge of Yang-Mills theory and the functional method, which are discussed briefly in Section 1. The bibliography - we make no pretense for it being complete - given at the end of these notes provides the student with material for further reading on these topics.

Some of the results given in Section 7, especially those pertaining to Mandelstam's prescription for light-cone gauge integrals, are new, having been derived after the lectures were given. These results, together with Section 8, the content of which became possible for discussion within the context of these notes only after the new results were obtained, are included here for completeness.

My gratitude to Michael Milgram is best expressed by saying that without his collaboration all of the work reported here would not have been done. I am thankful to George Leibbrandt for his help during the early phase of work and for his continuing interest. I thank Kuo-Lung Chang for the invitation to National Taiwan University and the Faculty of the Physics Department for its hospitality during my stay there, where these notes were first drafted. I thank the Physics Departments of Cheng-Kung University (Tainan), Zhengzhou University, Wuhan University, Hiroshima University and Tokyo University (Komaba), and also the Institute of High Energy Physics (Beijing), the Institute for Theoretical Physics (Beijing) and the Institute for Fundamental Research (Kyoto), where parts of these lectures were given, for hospitality. Last but not least, I thank Margaret Carey for carefully preparing these notes.

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H.C. Lee, Chalk River, 1984

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1. Yang-Mills Theory, Gauge Fixing and Ghost-Free Axial Gauges

In the functional method, the sourceless (J=0) generating functional for a Yang-Mills theory with fields A^a_μ is given by

$$Z[J=0] = Z[0] = \int [dA] e^{iS[A]} \tag{1.1}$$

where

$$S = \int d^4x\, \mathcal{L} \tag{1.2}$$

is the action and

$$\mathcal{L} = -\frac{1}{4} \underline{F}_{\mu\nu} \cdot \underline{F}^{\mu\nu} \tag{1.3}$$

is the Lagrangian density, and the field tensor is

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu + g(\underline{A}_\mu \times \underline{A}_\nu) \;. \tag{1.4}$$

The gauge field $\underline{A}_\mu$ transforms as a vector of the gauge group G. The components of such vectors will be labeled by the indices a,b,···. The scalar product and the cross product in (3) and (4) are defined respectively as

$$\underline{A} \cdot \underline{B} = A^a B^a \;, \; (\underline{A} \times \underline{B})^a = f^{abc} A^b B^c \tag{1.5}$$

where f^{abc} are structure constants of G. The symbol $\int[dA]$, or the path integral, in (1) is meant to integrate over all possible values of each gauge field $A^a_\mu(x)$ at each space-time point x. In the following, we will often drop all labels of the gauge field and simply express it as A.

The group G is defined by the set of "gauge" transformations leaving the action S invariant. Let h be an element of G, $h \in G$. Then under the action of h,

$$A \to A^h$$

$$S[A] \to S[A^h] = S[A] \quad . \tag{1.6}$$

The transformations are gauge transformations, which are local because h is a function of space-time. Clearly if the number of elements in G is N then S is N-fold degenerate. For Yang-Mills theories G is a Lie group which, being continuous, has an infinite number of elements. Thus S has an infinite degeneracy. It follows that the generating function Z[0] in (1) is not well-defined, since it contains an infinite factor proportional to

$$\Big(\sum_{h \in G} \int [dA^h] \Big) \equiv \int [dA] \int dh \tag{1.7}$$

As we shall see later, a symptom associated with this infinite degeneracy is that the propagator derived from (2) will be singular.

In the path integral method, the infinite gauge degeneracy is removed by imposing on the integral a functional constraint

$$F[A] = 0 \tag{1.8}$$

that breaks the gauge invariance, thus insuring that each infinite set of gauge equivalent paths will be integrated over only once. This is a method first used by Faddeev and Popov.[1] Here we follow Lee.[2]

Define the functional

$$\Delta_F^{-1}[A] \equiv \int dh \;\; \delta(F[A^h]) \tag{1.9}$$

where $\int dh$ integrates over the group space for each A at each space-time point. It is clear that Δ_F is invariant under the transformation

$A \to A^{h'}$. We now insert the factor

$$1 = \Delta_F[A] \int dh \ \delta(F[A^h]) \tag{1.10}$$

into the right-hand-side of (1) to obtain

$$Z[0] = \int [dA] \Delta_F[A] \left\{ \int dh \ \delta(F[A^h]) \right\} e^{iS[A]} \ . \tag{1.11}$$

Because $\int[dA]$ integrates over all possible values of A, including those covered by gauge transformations, and because $\int[dA]$, $\Delta_F[A]$ and $S[A]$ are all gauge invariant, we may change variable

$$A \to A^{h^{-1}} \tag{1.12}$$

and rewrite (11) as

$$Z[0] = \int [dA] \ \Delta_F[A] \ \delta(F[A]) \ e^{iS[A]} \ \int dh \tag{1.13}$$

to isolate the infinite normalization $\int dh$ mentioned earlier. We now redefine Z[0] by removing from it this infinite factor, so that

$$Z[0] \equiv \int [dA] \Delta_F[A] \delta(F[A]) e^{iS[A]} \tag{1.14}$$

is now well-defined; in a manner dictated by F[A], the integral takes only one path among each set of gauge equivalent paths. This equation will not be suitable for computation in perturbation theory until the two factors $\Delta_F[A]$ and $\delta(F[A])$ are exponentiated.

For the first factor, we recall the identity

$$\int dx \ \delta(f(x)) = \left(\frac{\partial f}{\partial x}\right)^{-1}_{f=0} \tag{1.15}$$

and write

$$\Delta_F^{-1}[A] = \int dh \;\; \delta(F[A^h])$$

$$= \prod_{x,a} \int dh \;\; \delta[F\{A^a(x)\}^h])$$

$$= \det{}^{-1} \left|\frac{\delta F[A]}{\delta h}\right|_{F=0}$$

$$= \det{}^{-1} \left|\frac{\delta F[A]}{\delta A} \; \frac{\delta A}{\delta h}\right|_{F=0}$$

$$\equiv \det{}^{-1}(M_F) \tag{1.16}$$

The matrix M_F has elements

$$\langle x,a\left|M_F\right|y,b\rangle = \delta^4(x-y) \; \frac{\partial F^a}{\partial A_\mu^c(x)} \; D_\mu^{cb}(x) \tag{1.17}$$

where

$$D_\mu^{cb}(x) = \delta^{cb}\partial_\mu - igf^{cba} \; A_\mu^a \tag{1.18}$$

is the covariant derivative associated with gauge transformations under G. Eq. (16) also reads

$$\Delta_F[A] = \det(M_F) \;. \tag{1.19}$$

But a determinant can be expressed as a path integral of anticommuting fields.[2] Let such fields be ξ and η; then one may write

$$\det(M_F) = \int[d\xi][d\eta]e^{iS_{ghost}} \tag{1.20}$$

$$S_{ghost}[A,\xi,\eta] \equiv \int d^4x \mathscr{L}_{ghost} = \int d^4x \;\; \xi_a(M_F)^{ab}\eta_b \tag{1.21}$$

ξ and η are called ghosts (fields) because they do not represent physical particles, but owe their existence purely to the constraint (8). Ghosts do not appear as external legs in any Feynman diagram representing a physical amplitude but do propagate as virtual particles. They interact with the gauge fields through the term $\xi M_F \eta$ in S_{ghost}.

We now turn to exponentiating the factor $\delta(F[A])$. The simple way, sufficient for our purpose, is to make use of the relation

$$\delta(F) \propto \lim_{\alpha \to 0} e^{-iF^2/2\alpha} \tag{1.22}$$

A more general result, replacing the constraint $\delta(F[A])$ by the weighted constraint

$$\int dc\ e^{-ic^2/2\alpha}\ \delta(F[A]-c) \tag{1.23}$$

where c is independent of A, yields

$$\delta(F-c) \to e^{-iF^2/2\alpha} \tag{1.24}$$

with α now being an arbitrary parameter (the gauge parameter), in particular not restricted in value to the limit $\alpha \to 0$. It is clear that the right-hand-side of (24) no longer implies the constraint $F = 0$. The constraint (22) is obviously a special case of (24) which does imply $F = 0$. Recall that the constraint (24) or (22) is imposed on each and every space-time point, so (24) suggest the gauge fixing action

$$S_{g.f.}[A] = - \int d^4x\ (F[A])^2/2\alpha\ . \tag{1.25}$$

Combining (14), (20), (24) and (25) then leads to the effective action

$$S_{eff}^{[A,\xi,\eta]} = S^{[A]} + S_{ghost}^{[A,\xi,\eta]} + S_{g.f.}[A] \tag{1.26a}$$

and the generating functional with a fixed gauge

$$Z[0] = \int [dA][d\xi][d\eta]\; e^{iS_{eff}[A,\xi,\eta]} \;. \tag{1.26b}$$

Finally, many formal relations are more easily derived from a generating functional with source $\underline{J}^{\mu}(x)$, defined as

$$Z[J] = \int [dA][d\xi][d\eta]\; e^{iS_{eff}[A,\xi,\eta,J]}, \tag{1.27a}$$

$$S_{eff}[A,\xi,\eta,J] = S_{eff}[A,\xi,\eta] + \int d^4x\; \underline{J}^{\mu}\cdot\underline{A}_{\mu} \tag{1.27b}$$

A class of widely used (Lorentz) covariant gauges is specified by

$$F[A] = \partial_{\mu}\,\underline{A}^{\mu} \tag{1.28}$$

When the condition F=0, realized in the limit $\alpha\to 0$ in (24), is chosen, this gauge is analogous to the Lorentz gauge in quantum electrodynamics. The ghost action associated with (28) is

$$S_{ghost} = \int d^4x\; \xi_a [\delta^{ab}\partial^{\mu}\partial_{\mu} - igf^{abc}(\partial^{\mu}A^c_{\mu} + A^c_{\mu}\partial^{\mu})]\eta_b \tag{1.29}$$

In the limit $\alpha \to 0$, $\partial^{\mu}A^c_{\mu} = 0$, so

$$\lim_{\alpha\to 0} S_{ghost} = \int d^4x\; \xi_a (\delta^{ab}\partial_{\mu} - igf^{abc}A^c_{\mu})\partial^{\mu}\eta_b \tag{1.30}$$

the ghosts are still coupled to the gauge field through the $\xi A_{\mu}\partial^{\mu}\eta$ term.

There is therefore no simplification when the limit $\alpha \to 0$ is chosen. The gauges fixed by (28) with α having various special values have been given names associated with their earliest proponents: $\alpha=0$ is the Landau gauge; $\alpha=1$ is the Feynman gauge; $\alpha=1/3$ is the Yennie gauge.

The axial gauges[3] form a class of gauges characterized by the usage of a constant vector n_μ. This singles out a special direction in space time, so such gauges are not Lorentz covariant. The simplest of the axial gauges has

$$F[A] = n_\mu A^\mu \qquad (1.31)$$

which has the ghost action

$$S_{ghost} = \int d^4x\ \xi_a [\delta^{ab} n^\mu \partial_\mu - igf^{abc}\ n^\mu A_\mu^c]\ \eta_b \qquad (1.32)$$

Significant in (32) is the absence, due to the fact that n_μ is a constant instead of an (derivative) operator, of a term corresponding to the $\xi A_\mu \partial^\mu \eta$ term in (29). Thus in the limit $\alpha \to 0$, $n_\mu A^\mu = 0$, so that

$$\lim_{\alpha \to 0} S_{ghost} = \int d^4x\ \xi_a n^\mu \partial_\mu \eta_a \qquad (1.33)$$

is independent of gauge fields. This means that for axial gauges in the limit $\alpha \to 0$ ghosts are decoupled from the rest of the theory, and the factor

$$\lim_{\alpha \to 0} \int [d\xi][d\eta] e^{iS_{ghost}} = \text{const.}$$

becomes just an insignificant normalization constant for the generating functional.

It is clear from the discussion above that for an axial gauge to be ghost-free there are two necessary conditions:

(i) The constraint must not involve any derivative operator acting on the gauge field;

(ii) The value for the gauge parameter must be taken in the limit $\alpha\to 0$.

An example for an axial gauge constraint containing the derivative and therefore not ghost-free even at $\alpha\to 0$ is[4]

$$F[A] = n_\mu \partial^\mu n_\nu \underline{A}^\nu \tag{1.34}$$

giving a ghost action

$$S_{ghost} = -\int d^4x \; [(n_\mu \partial^\mu \xi_a)(n_\nu \partial^\nu \eta_a) + g f^{abc} A^c_\mu \xi_a (\partial^\mu \eta_b)] \; . \tag{1.35}$$

In the rest of these lectures, by axial gauge we shall mean a gauge constrained by (31), for which the limit $\alpha\to 0$ will always be taken. Thus

$$\begin{aligned} S^{axial}_{eff}[A] &= \lim_{\alpha\to 0} \int d^4x \; [-\tfrac{1}{4}\, \underline{F}_{\mu\nu}\cdot \underline{F}^{\mu\nu} - \tfrac{1}{2\alpha}\,(n_\mu \cdot \underline{A}^\mu)^2] \\ &= \lim_{\alpha\to 0} \{\tfrac{1}{2} \int d^4x \; [\underline{A}^\mu \cdot (\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu - \tfrac{1}{\alpha}\, n_\mu n_\nu) \underline{A}^\nu \\ &\quad - g(\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu)\cdot(\underline{A}^\mu \times \underline{A}^\nu) - \tfrac{g^2}{2}\,(\underline{A}_\mu \times \underline{A}_\nu)^2]\} \end{aligned} \tag{1.36}$$

from which one can read off the kinetic energy, or the coefficient of the term quadratic in A, in the momentum representation,

$$\Pi^{(0)ab}_{\mu\nu} \equiv \delta^{ab}\, \Pi^{(0)}_{\mu\nu} = -i\,\delta^{ab}\left(p^2 \delta_{\mu\nu} - p_\mu p_\nu + \tfrac{1}{\alpha}\, n_\mu n_\nu\right) \tag{1.37}$$

and the three and four-vertices

$$_3\Gamma^{(0)abc}_{\lambda\mu\nu} = gf^{abc}[\delta_{\lambda\mu}(p-q)_\nu + \delta_{\mu\nu}(q-r)_\lambda + \delta_{\nu\lambda}(r-p)_\mu] \qquad (1.38)$$

$$_4\Gamma^{(0)abcd}_{\mu\nu\rho\sigma} = -ig^2[f^{abe}f^{cde}(\delta_{\mu\rho}\delta_{\nu\sigma} - \sigma_{\mu\sigma}\delta_{\nu\rho}) + \binom{b\leftrightarrow c}{\nu\leftrightarrow\rho} + \binom{b\leftrightarrow d}{\nu\leftrightarrow\sigma}] \qquad (1.39)$$

The reciprocal of $\Pi^{(0)ab}_{\mu\nu}$ gives the free propagator

$$\Delta^{(0)ab}_{\mu\nu} \equiv \delta^{ab}\ \Delta^{(0)}_{\mu\nu} = \frac{i\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n} + (n^2 - \alpha p^2)\frac{p_\mu p_\nu}{(p\cdot n)^2}\right] \qquad (1.40)$$

As is well known, it is much easier to evaluate Feynman integrals and to discuss most formal properties of gauge theories in Euclidean space. Unless otherwise mentioned, we shall work in this space in these notes. In practice working in the Euclidean space means replacing the metric $g_{\mu\nu}$ by the Euclidean metric

$$g_{\mu\nu} \to \delta_{\mu\nu} = (1,1,1,1) \qquad (1.41)$$

For Feynman integrals, Minkowski space can be reached by analytic continuation after the integrals have been evaluated.

The singularity in $\Delta^{(0)}_{\mu\nu}$ in the limit $\alpha\to\infty$ is directly related to the divergence of the generating functional for a gauge invariant action. For in this limit the gauge fixing action vanishes. Another way of recognizing this singularity is to observe that the matrix

$$p^2 g_{\mu\nu} - p_\mu p_\nu$$

has a null determinant.

In perturbation theory the propagator (40) and the three and four-vertices (38) and (39) are the only quantities needed for computation.

Among these quantities only $\Delta^{(0)}_{\mu\nu}$ depends on α, being finite at $\alpha=0$. Therefore the limit $\alpha\to 0$ can be taken at this stage, thus making the theory ghost free. The fact that $\Pi^{(0)}_{\mu\nu}$ diverges in this limit may appear worrisome. Later (see §6) we shall show that the term $\frac{1}{\alpha}\, n_\mu n_\nu$ in $\Pi^{(0)}_{\mu\nu}$ is totally decoupled from the rest of the theory. Whether it diverges or not is therefore of no significance.

For completeness we give the propagator in the covariant gauges defined by the constraint (28):

$$\Delta^{(0)\,\text{covariant gauge}}_{\mu\nu} = i\,\frac{\delta^{ab}}{p^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right] \qquad (1.42)$$

2. Regularization and Dimensional Regularization

2.1 Need for Regularization

The class of processes involving the creation and annihilation of virtual particles is what sets quantum field theory apart from the classical theory. A typical such process is the vacuum polarization represented by the Feynman diagram in Fig. 1

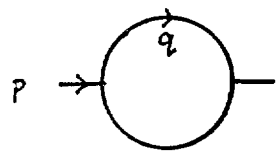

Figure 1

where the arrows represent the flow of momenta. Using the propagator and the three-vertex given in (1.40) and (1.38) respectively to compute this diagram (see (6.8)) entails the evaluation of the following integral, among others,

$$I(p) = \int d^4q \frac{1}{(p-q)^2 q^2} \tag{2.1}$$

Simple power counting shows that I(p) is (logarithmically) divergent at $q \to \infty$, or ultraviolet (UV) divergent, rendering the integral meaningless. One way to interpret the existence of such divergences in quantum field theories is to think of such theories as being incomplete in the region of infinitely large momentum, or at distances very close to zero. A procedure known as renormalization has been developed to control UV divergences in quantum field theories in such a way that physical occurrences at finite momenta are precisely described independently of what may happen at infinite momentum. The first step in this procedure is to regularize, or regulate, divergent integrals such as (1).

Loosely speaking, to regulate a divergent integral is to isolate the infinite and regular parts of the integral in a well-defined way. It is clear that such a separation is not unique, for if

$$I = \infty + a$$

is a separation, then

$$I = (\infty + b) + (a - b)$$

is also a separation. This means that there can be more than one viable regularization method. On the other hand, if a renormalization program is to be meaningful it must give results describing physical occurrences that are independent of the regularization method.

One of the oldest and sometimes still used regularization methods is due to Pauli & Villars.[5] It entails the replacement

$$\frac{1}{(p-q)^2} \to \frac{1}{(p-q)^2} - \frac{1}{(p-q)^2+M^2} \tag{2.2}$$

for some or all factors in the denominator of an integral so that the integral (1) may be formally defined in the limit

$$I(p) \overset{\text{def}}{=} \lim_{M\to\infty} \int d^4q \left[\frac{1}{(p-q)^2} - \frac{1}{(p-q)^2+M^2}\right]\left[\frac{1}{q^2} - \frac{1}{q^2+M^2}\right] \tag{2.3}$$

The integral on the right-hand-side can be rewritten as

$$M^4 \int d^4q \frac{1}{(p-q)^2[(p-q)^2+M^2]q^2(q^2+M^2)} \tag{2.4}$$

which manifestly does not diverge as $q\to\infty$; it is finite for all finite values of M.

Another regularization method is the cut-off method

$$\int d^4q\ K(p,q) \to \lim_{\Lambda\to\infty} \int_{q^2<\Lambda^2} d^4q\ K(p,q). \tag{2.5}$$

Such methods, although useful under certain circumstances, have shortcomings arising from their undesirable algebraic and/or analytic properties. For example distributivity

$$\int d^4q(A+B) = \int d^4q\ A + \int d^4q\ B \tag{2.6}$$

and partial fraction

$$\int d^4q \frac{1}{(A+a)(A+b)} = \frac{1}{a-b} \int d^4q \left(\frac{1}{A+b} - \frac{1}{A+a}\right) \tag{2.7}$$

are sometimes lost in the Pauli-Villars method and the shift operation, or translational invariance,

$$\int d^4q\ A(q) = \int d^4q\ A(q+q_0) \tag{2.8}$$

is lost to the cut-off method. In the above equations A and B are functions of q, a and b are constants and q_0 is a fixed momentum. An even more severe defect suffered by both methods is that they do not preserve gauge invariance. Technically the methods are also very cumbersome. As a general rule the evaluation of a "massive integral" such as (4) is always considerably more tedious than that of a "massless integral" such as (1).

2.2. Dimensional Regularization

A very powerful method, known as dimensional regularization,[4] based on the principle of analytic continuation, exploits the possibility of defining integrals in a continuous dimensional space. In dimensional regularization (dim. reg.), instead of evaluating an integral such as

$$I[S;2] \equiv \int d^4q \; S(q) \;, \tag{2.9}$$

one considers as a function of the continuous (possibly complex) variable ω the integral

$$I[S;\omega] \equiv \int d^{2\omega}q \; S(q) \tag{2.10}$$

and defines

$$I[S;2] \stackrel{\text{def}}{\equiv} \lim_{\omega\to 2+\varepsilon} I[S;\omega] \tag{2.11}$$

The method is useful because divergent Feynman integrals defined as in (11) become functions having poles at $\omega=2$ in the complex ω-plane, which therefore have well-defined mathematical properties.

Because dim. reg. is an analytic method, it has very good algebraic properties. In particular the method admits

commutivity: $A \int B = \int AB$ (2.12a)

distributivity: $\int(A + B) = \int A + \int B$ (2.12b)

associativity: $\int(ABC) = \int(AB)C + \int A(BC)$ (2.12c)

shift operation: $\int d^{2\omega}q \; A(q) = \int d^{2\omega}q \; A(q+q_0)$ (2.12d)

The following illustrates how all of the above can be exploited to simplify a computation.

$$p_\mu \int dq\; q_\mu/[(p-q)^2 q^2] = \int dq\; p\cdot q/[\cdots] \qquad \text{(comm.)}$$

$$= \frac{1}{2} \int dq[p^2+q^2 - (p-q)^2]/[\cdots]$$

$$= \frac{1}{2} \left\{ \int dq\; p^2/[\cdots] + \int dq\; q^2/[\cdots] - \int dq(p-q)^2/[\cdots] \right\} \qquad \text{(dist.)}$$

$$= \frac{1}{2}\, p^2 \int dq/(p-q^2)q^2 + \frac{1}{2} \int dq/(p-q)^2 - \frac{1}{2} \int dq/q^2 \qquad \text{(ass.)}$$

$$= (p^2/2) \int dq/(p-q)^2 q^2 \qquad \text{(shift)} \qquad (2.13)$$

Hereafter, where there is no risk for confusion, we will often use the shorthand $\int dq$ for $\int d^{2\omega}q$. The last line in (13), having a scalar integrand, is easier to compute than the original integral with a vectorial integrand. The manipulations employed in (13), although standard for finite and well-defined integrals, are in general suspect for divergent integrals. In particular they are not allowed when $\omega=2$, nor are they proper for the nonanalytic Pauli-Villars and cut-off regularizations described earlier.

A crucial property of dim. reg. is that it preserves the gauge invariance of gauge theories. This and reasons given earlier concerning its superior algebraic properties explain why the method has been used exclusively in the proof of the renormalizability of nonAbelian gauge theories.

Powerful as it is, dim. reg. still has some deficiencies:

- Formally the method cannot regulate 'tadpole' integrals (see below);

- The method does not distinguish ultraviolet (UV) from infrared (IR) divergences;
- The method is not powerful enough to regulate some integrals in the axial gauges.

2.3 Tadpoles

Tadpoles are Feynman diagrams containing loops but having only one vertex connected to external legs. Conservation of momentum then dictates that the integrand for the corresponding loop integral cannot depend on any external momentum. The simplest tadpole is a 1-loop diagram with one external leg, as shown in Fig. 2.

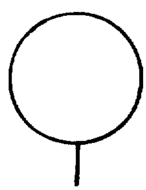

Figure 2. A tadpole.

This diagram has the simplest integral for Yang-Mills theory in the Feynman gauge (α = 1)

$$\int d^4q \, \frac{1}{q^2} \stackrel{\text{def}}{=} \lim_{\omega \to 2+\varepsilon} \int d^{2\omega}q \, \frac{1}{q^2} \equiv I(\omega) \tag{2.14}$$

As will be described in detail in §3, the way to proceed is to first identify a region in the ω plane in which $I(\omega)$ is well-defined, evaluate the integral in that region, and then analytically continue the result to the limit $\omega \to 2$. Now simple power counting tells us that $I(\omega)$ is

- UV divergent (at $q^2 \to \infty$) when $\mathrm{Re}(\omega) \geq 1$,
- IR divergent (at $q^2 \to 0$) when $\mathrm{Re}(\omega) \leq 1$,

so no region in the ω-plane exists in which $I(\omega)$ is regular. Therefore dim. reg. cannot be employed to regulate $I(\omega)$. This result is general: the set of integrals

$$I_N = \int d^4q \, (q^2)^N$$

cannot be regulated by dim. reg. for any N. Therefore, if dim. reg. is used as a regularization method, a supplementary ansatz must be given to deal with tadpoles. Conventionally the definition

$$\int d^{2\omega}q \, (q^2)^N \stackrel{\text{def}}{=} 0 \qquad \text{(dim. reg.)} \tag{2.15}$$

has been adopted. In §4 we shall show rigorously that this definition, although surprising at first sight, is an appropriate one. A rigorous theory of similar integrals is well known to mathematicians.[30]

2.4 Infrared and Ultraviolet Divergences

In theories with massless particles Feynman integrals may be IR divergent as well as UV divergent. These divergences arise for different reasons and are to be treated differently. Infinities of the UV type are to be absorbed into renormalization constants for wavefunctions and coupling constants whereas IR infinities are to be cancelled by their counterparts arising from radiation of real, low energy gauge bosons. For this reason it is sometimes desirable to separate the two types of infinities. In dim. reg., all infinities arising from integration manifest themselves as identical poles having the form $1/(\omega-2)$ in the ω-plane (2ω is the generalized number of dimensions). The separation of UV from IR singularities is therefore generally not straightforward.

A commonly used technique for isolating IR singularities is to assign masses m_i to particles with which the singularities are associated. In this way UV singularities remain as poles in the ω-plane whereas IR singularities are converted to logarithmic singularities $\ln m_i$ in the limit $m_i \to 0$. While effective, this technique replaces massless integrals by massive ones thus invariably making them more difficult to evaluate. Because giving masses to massless gauge bosons also destroys the gauge invariance of the original theory, Ward-Takahashi identities are often lost as a tool for checking the calculation.

The inseparability of UV and IR singularities in dim. reg. is intimately related to the value assigned to tadpole integrals in that method. Later in §5 we shall show that in many cases the vanishing of tadpoles result from the cancellation between the two types of singularities.

2.5 Axial Gauge Singularity and the Principal-Value Prescription

In axial gauges the factor $p \cdot n$ appearing in the denominators in two of the terms for the propagator (1.40) gives rise to a third kind of singularity in some Feynman integrals. An example of such an integral is

$$\int d^4q \, \frac{1}{(p-q)^2 (q \cdot n)^2} \tag{2.16}$$

which, in addition to being UV divergent, has an additional singularity associated with the possibility that the quantity $q \cdot n$ vanishes.

Unlike UV and IR singularities, the axial gauge (AG) singularity is purely an artifact of gauge fixing and is void of physical meaning. It is nevertheless very real from the mathematical point of view and must be dealt with if any computation is to be done in an axial gauge.

Until recently the most successful and widely used method to handle the AG singularity was the principal-value prescription,[7] in which negative powers of the factor q•n are defined as the limit

$$(q\cdot n)^{-N} \stackrel{\text{def}}{=} \frac{1}{2} \lim_{\eta\to 0} \left[(q\cdot n + i\eta)^{-N} + (q\cdot n - i\eta)^{-N}\right] . \tag{2.17}$$

Combined with dim. reg., integrals involving such factors are then defined as

$$\int d^4q\ S(q)(q\cdot n)^{-N} \stackrel{\text{def}}{=}$$

$$= \frac{1}{(N-1)!} \lim_{\eta\to 0} \mathrm{Re}\left(\frac{\partial}{\partial\eta}\right)^{N-1} \left\{\lim_{\omega\to 2+\varepsilon} \int d^{2\omega}q\ S(q)\ \frac{q\cdot n - i\eta}{(q\cdot n)^2+\eta^2}\right\} . \tag{2.18}$$

The evaluation of the right-hand-side, involving two limiting processes, is usually a tricky and difficult task. We shall give an example of it later.

3. Generalization and Analytic Continuation

We shall now introduce a new regularization method which we call analytic regularization (an. reg.). In essence it is a generalization of dim. reg. designed to remove the shortcomings of that method mentioned earlier. The term analytic regulariztion has been used for some older methods[8] employing techniques similar to those employed here for the new method. The older methods were abandoned partly because they were incomplete, and more importantly because the analytic technique employed in the method was given an incorrect physical interpretation. We shall return to discuss the old method in a more appropriate context later.

To understand an. reg. properly it is important to have a clear understanding of the two important steps used in the method, generalization and analytic continuation, which will be discussed below. Luckily, both are well established subjects in the theory of functions, relieving us of any need for detail and rigor in our treatment.

Consider a function f_i formally defined on a set S of discrete points x_i which can be divided into two subsets, S_A and S_B,

$$S = S_A \cup S_B$$

such that f_i is well-defined if x_i belongs to S_A but is ill-defined if x_i belongs to S_B.

An example for such a function is the set of integrals

$$f_N \equiv \int d^{2N}q \, \frac{1}{(p-q)^2 \, q^2} \tag{3.1}$$

formally defined over the 2N-dimension integration (Euclidean) space.

The set S of points in this case contain the set of half integers 1/2, 1, $\cdots$. In writing f_N the notion that the right-hand-side must also be a function of p^2 is suppressed. Upon inspection one finds that f_N is UV divergent when $N \geq 2$, is IR divergent when $N \leq 1$, and is regular only when $N = 3/2$. That is,

$$\begin{aligned} S_A &= \{3/2\} \\ S_B &= \{1/2,\ 1,\ 2,\ 5/2,\ \cdots\}\ . \end{aligned} \tag{3.2}$$

Thus, as it stands, f_N is meaningful only when $N = 3/2$. To make sense of f_N with $N \in S_B$ the integral in (1) must be regularized.

This can often be accomplished by first generalizing the definition of the original function. Instead of considering f_i on the set $x_i \in S$, we consider the function $f(x)$ formally defined for the continuous variable (which may be complex) x in the region R. The generalization is therefore

$$\begin{aligned} &f_i \rightarrow f(x) \\ &x_i(\text{points}) \rightarrow x\ (\text{continuous variable}) \\ &S\ (\text{set}) \rightarrow R\ (\text{region}) \end{aligned} \tag{3.3}$$

Obviously, if the generalized function is to have anything to do with the original one, R must contain S,

$$S \subset R. \tag{3.4}$$

The mapping (3) satisfying (4) is given schematically in Fig. 3 where R is shown to contain at least two regions R_A and R_B satisfying

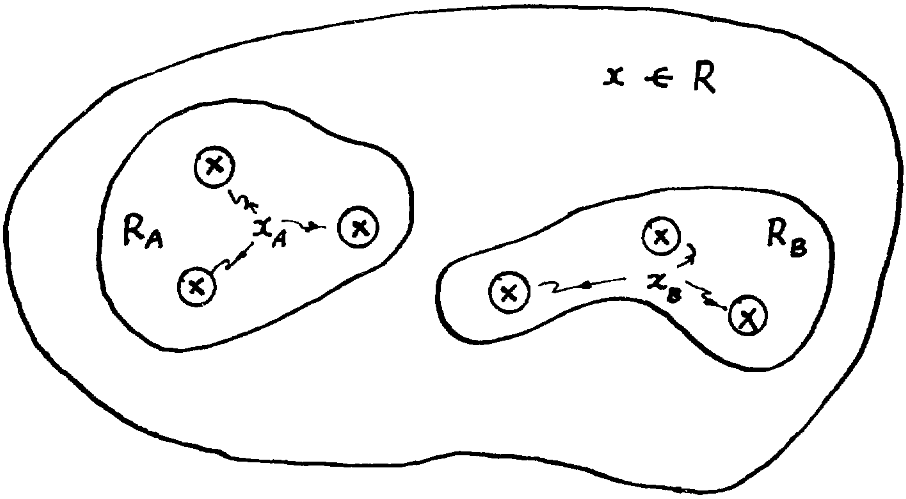

Figure 3

$$R \supset R_A \oplus R_B, \quad R_A \supset S_A, \quad R_B \supset S_B \tag{3.5}$$

such that f(x) is well defined if $x \in R_A$ and ill-defined if $x \in R_B$. Furthermore, for the points $x_i \in R$, $f(x_i)$ must be formally identical to f_i

$$f(x_i) \equiv f_i \ , \qquad x_i \in S \subset R. \tag{3.6}$$

It is clear that (6) does not uniquely define f(x), since the original definition of f_i says nothing about points not belonging to S. It follows that for any set of f_i infinitely many generalizations are possible. This also explains why there have been so many regularization methods for divergent Feynman integrals.

Returning now to our example (1), one of the possible generalizations for f_N is precisely that used in dimension regularization, where the set of half-integers N is generalized into the continuous complex variable ω:

$$f_N \to f(\omega) \equiv \int d^{2\omega}q \, \frac{1}{(p-q)^2 \, q^2} \, , \qquad N \to \omega \quad C^1 \tag{3.7}$$

and the region to which ω belongs is the 1-dimensional complex space C^1. Since normally integration is defined only for integer-dimension spaces, one must specify what is meant by the notation $\int d^{2\omega}q$ in (7). The definition given to it in dim. reg., which is also the one we shall adopt, is specified via the generalized gaussian integral

$$\int d^{2\omega}q \; e^{-aq^2} \overset{\mathrm{def}}{=} (\pi/a)^{\omega} \, . \tag{3.8}$$

and this is sufficient for one to do all Feynman integrals, as far as the portion relating to continuous dimensions is concerned.

By power counting, the integral is well-defined at least in the line section $1 < \mathrm{Re}(\omega) < 2$. It is therefore meaningful to compute the integral. The result, in conjunction with (8) (see a later section for technical detail), is

$$\int d^{2\omega}q \, \frac{1}{(p-q)^2 q^2} = \pi^{\omega}(p^2)^{\omega-2}\,[\Gamma(\omega-1)]^2\,\Gamma(2-\omega)/\Gamma(2\omega-2)$$

$$\equiv g(\omega) \, , \qquad\qquad 1 < \mathrm{Re}(\omega) < 2 \, . \tag{3.9}$$

Note however that the right-hand-side of the integral, $g(\omega)$, is well-defined even beyond the region specified in (9). In fact $g(\omega)$ is regular everywhere in C^1 except when ω is equal to any integer. It is important to realize that formally $f(\omega)$ is not identical to $g(\omega)$. For power counting shows that $f(\omega)$ is ill-defined at least when $\mathrm{Re}(\omega) \le 1$ and $2 \le \mathrm{Re}(\omega)$, while from the analytic property of the gamma function $\Gamma(z)$ (poles for nonpositive integral values for z) $g(\omega)$ is regular everywhere for $\omega \quad C^1$ except when $\omega = 0, \pm 1, \pm 2, \cdots$, in which case it has single poles; $g(\omega)$ is well-defined everywhere in C^1. At least in the region $1 < \text{Re}(\omega) < 2$, $f(\omega)$ is well-defined and equal to $g(\omega)$. Therefore, according to the principle of analytic continuation, the region in which $g(\omega)$ may be used to represent $f(\omega)$ can be extended to cover the whole region in which $g(\omega)$ is well-defined, that is for all $\omega \in C^1$. Thus, $g(\omega)$ is a representation for $f(\omega)$ for $\omega \in C^1$. With this understanding, the distinction between $f(\omega)$ and $g(\omega)$ may now be forgotten, and the restriction given in (9) on the region of validity can be neglected.

After the two-step process of generalization and analytic continuation, $g(\omega)$ can now be used as a representation for the original sometimes ill-defined set of integrals f_N in (1), as follows:

$$f_N \stackrel{\text{def}}{=} \lim_{\omega \to N} g(\omega) \tag{3.10}$$

The right-hand-side has a pole at $\omega = 1$ reflecting the IR divergence of f_1 and poles at $\omega = 2, 3, \cdots$ reflecting UV divergences of f_2, f_3, $\cdots$. However, the representation is regular at all positive half-integer values for ω even as f_N at $N = 1/2$ and $N = 5/2, 7/2, \cdots$ are ill-defined. This is important: a representaticn derived via generalization and analytic continuation of a ill-defined function need not be singular; it can be regular! The right-hand-side of (10) has at most pole singularities, which have well-defined analytic properties, it is therefore said to be a regularization of the left-hand-side.

The relations between f_N, $f(\omega)$ and $g(\omega)$ are summarized in the following diagram.

$$f_N \equiv \int d^{2N}q \, \frac{1}{(p-q)^2 q^2}, \qquad N = 1/2,\ 1,\ 3/2,\ \cdots$$

← Regularization

↓ Generalization

$$f(\omega) \equiv \int d^{2\omega}q \, \frac{1}{(p-q)^2 q^2}, \qquad \omega \in C^1$$

$$= \frac{\pi^{\omega}(p^2)^{\omega-2}[\Gamma(\omega-1)]^2\Gamma(2-\omega)}{\Gamma(2\omega-2)}, \qquad 1 < \mathrm{Re}(\omega) < 2$$

↓ Analytic Continuation

$$g(\omega) \equiv \frac{\pi^{\omega}(p^2)^{\omega-2}[\Gamma(\omega-1)]^2\Gamma(2-\omega)}{\Gamma(2\omega-2)}, \qquad \omega \in C^1$$

$\lim_{\omega \to N}$

Figure 4. Regularization of the Function $\int d^{2N}q \, \frac{1}{(p-q)^2 q^2}$

It is clear that in the analytic method the representation function $g(\omega)$ plays a pivotal role. For without it the crucial analytic continuation which allows one to avoid the singularities in the original function cannot be carried out, leaving the regularization process incomplete. However, for instances where only the formal existence of a representation is needed, the key point then becomes whether there is a region for the generalized variable in which the generalized function $f(\omega)$ is well-defined. For if there exists such a region, then the value of the integral must be an explicit (albeit perhaps unknown) function of ω in that region. One can then identify this function with $g(\omega)$ and assume that it is amenable to analysis and analytic continuation.

The example discussed above gives the basis for dim. reg. In the following we give another example in which a certain type of singularity is regulated by generalizing the exponent of an expression. The method used in this example later will be exploited to regulate the axial gauge singularity that was mentioned in section 2.3; it is the starting point from which we develop our analytic regularization.

Consider the set of integrals

$$f_N \equiv \int_0^1 dt \, \frac{1}{(t-s)^N} \qquad \begin{matrix} N = \cdots,-1,0,1,\cdots \\ 0 < s < 1, \end{matrix} \tag{3.11}$$

which is ill-defined when $N = 1,2,\cdots$. For $N = 1$, f_1 is regulated by the well-known prescription

$$f_1 \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \left(\int_0^{s-\varepsilon} + \int_{s+\varepsilon}^1 \right) \frac{dt}{t-s} = \ln\left(\frac{1-s}{s}\right) \tag{3.12}$$

This jump-over-the-singularity ansatz is sometimes called the principal-value prescription but it actually cannot be extended to cases for $N \geq 2$. The correct principal-value prescription is

$$f_N \overset{\text{def}}{=} \frac{1}{2} \lim_{\eta \to 0} \int_0^1 dt \, \left\{ \frac{1}{[(t-s)+i\eta]^N} + \text{c.c.} \right\} \tag{3.13}$$

leading to the result

$$f_N = \lim_{\eta\to 0} \frac{1}{(N-1)!} \mathrm{Re}\left(i \frac{\partial}{\partial\eta}\right)^{N-1} \ell n \left(\frac{1-s+i\eta}{-s+i\eta}\right) \tag{3.14}$$

from which one obtains

$$f_1 = \ell n \left(\frac{1-s}{s}\right) \qquad \text{(as before)}$$

$$f_2 = -\left(\frac{1}{1-s} + \frac{1}{s}\right) \tag{3.15}$$

and so on. Although the computation involved in (14) is elementary it is nevertheless clear that evaluating f_N for large N can be very time-consuming.

We now do it the analytic way. We generalize f_N into

$$f(\nu) \equiv \int_0^1 dt(t-s)^\nu , \qquad \nu \in C^1 \tag{3.16}$$

For $\mathrm{Re}(\nu) > -1$, the left-hand-side of (16) is well-defined, so

$$f(\nu) = \frac{1}{\nu+1} \left[(1-s)^{1+\nu} - (-s)^{1+\nu}\right], \qquad \mathrm{Re}(\nu) > -1 \tag{3.17}$$

The right-hand-side is however regular in the whole C^1 space, including in the limit $\nu \to -1$, when it is equal to $\ell n\left(\frac{1-s}{s}\right)$ (recall the relation $\lim_{\varepsilon\to 0} x^\varepsilon = 1 + \varepsilon \ell n(x) + O(\varepsilon^2)$), agreeing with (12). Therefore by analytic continuation,

$$g(\nu) \equiv \frac{1}{\nu+1} \left[(1-s)^{1+\nu} - (-s)^{1+\nu}\right], \qquad \nu \in C^1 \tag{3.18}$$

is a representation of $f(\nu)$ in all of C^1. The analytic regularization for f_N is therefore

$$f_N \overset{\text{def}}{=} \lim_{\nu \to -N} g(\nu) \tag{3.19}$$

so that

$$f_1 = \ell n \left(\frac{1-s}{s}\right) ,$$

$$f_N = \frac{1}{1-N} \left[(1-s)^{1-N} + (-)^N s^{1-N}\right], \qquad N = \cdots -1,0,2,3,\cdots \tag{3.20}$$

The reader can verify for himself that (14) and (20) actually give identical results. He will also find that the time needed to evaluate (14) increases sharply with N, whereas the evaluation of (20) is trivial. Note that in both cases the regulated f_N is not only well-defined but also regular for all values of (integer) N, contrary to what the original definition may suggest. Here we see a pattern that will emerge again later: (a) the principal-value prescription and analytic regularization give identical results for apparent singularities of the type contained in f_N of (11); (b) the regulated function is regular.

The two-step process regulating f_N analytically is shown in Figure 5.

$$f_N \equiv \int_0^1 dt \, \frac{1}{(t-s)^N} \, , \qquad N = \cdots, -1, 0, 1, \cdots$$

← Regularization

Generalization

$$f(\nu) \equiv \int_0^1 dt \, (t-s)^{\nu} \, , \qquad \nu \in C^1$$

$$= \frac{1}{1+\nu} \left[(1-s)^{1+\nu} - (-s)^{1+\nu} \right] \, , \quad \mathrm{Re}(\nu) > -1$$

Analytic Continuation

$$g(\nu) \equiv \frac{1}{1+\nu} \left[(1-s)^{1+\nu} - (-s)^{1+\nu} \right] \, , \qquad \nu \in C^1$$

$\lim_{\nu \to -N}$

Figure 5. Regularization of $\int_0^1 dt \, \frac{1}{(t-s)^N}$

4. Analytic Regularization of the Class of Two-Point Integrals

In the last section we have demonstrated the complete analogy between dim. reg. that was used to regularize (3.1) and the regulariztion of (3.11). Each is but a special application of the analytic technique based on the two-step process of generalization and analytic continuation. We now apply this method, which we shall call analytic regularization, to regulate a whole class of two-point integrals - by an n-point integral we mean an integral with n-1 external momenta. The class of integrals is defined by

$$F(K,M,N,s) \equiv \int d^4q \; [(p-q)^2]^K \, (q^2)^M (q\cdot n)^{2N+s}, \quad s = 0 \text{ or } 1, \qquad K,M,N \text{ integers} \tag{4.1}$$

and the generalized form of the class is

$$S(\omega,\kappa,\mu,\nu,s) \equiv \int d^{2\omega}q\,[(p-q)^2]^{\kappa}(q^2)^{\mu}(q\cdot n)^{2\nu+s}, \qquad \omega,\kappa,\mu,\nu \in C^1. \tag{4.2}$$

It is clear that when $\nu = s = 0$, κ and μ = integers and $\omega = 2$, S reduces to (two-point) Feynman integrals in the covariant gauges; when κ, μ, $2\nu+s$ = integer and $\omega = 2$, S reduces to integrals in the axial gauge and when $\kappa=0$, μ, ν integer and $\omega=2$, S reduces to tadpole integrals. The feature distinguishing (2) from a generalized integral in dim. reg. is the generalized continuous exponents κ, μ and ν. The obvious motivation for such an extended generalization is that if a representation is found for (2), then the whole class of two-point Feynman integrals, in whatever gauge, can be simply evaluated by substitution. Later we shall also see that by seeking a more general representation, the three shortfalls of dim. reg. discussed in §2 concerning tadpoles, the separation of IR and UV singularities and the axial gauge singularity are all avoided.

We first examine the analytic properties of the S-integral and find it to be

(i)	UV div.	when	$\mathrm{Re}(\omega+\kappa+\mu+\nu) \geq 0$	
(ii)	IR div. (at p-q=0) when	when	$\mathrm{Re}(\omega+\kappa) \leq 0$	
(iii)	IR div. (at q=0) when	when	$\mathrm{Re}(\omega+\mu+\nu+s) \leq 0$	
(iv)	Axial-gauge singular	when	$\mathrm{Re}(\nu+s) \leq -1/2$	(4.3)

A representation for S exists if there is at least one region in $C^4 \times Z_2$ (the space in which $\{\omega, \kappa, \mu, \nu, s\}$ lives) where the integral exists. One such region is the neighbourhood of the point $\{\omega, \kappa, \mu, \nu, s\} = \{2,-1,-1,-1,1\}$. For at this point

$$\omega + \kappa + \mu + \nu = -1 < 0$$

$$\omega + \kappa = 1 > 0$$

$$\omega + \mu + \nu + s = 1 > 0$$

$$2\nu + s = 0 > -1/2$$

which is outside all the regions in which S is ill-defined. Therefore the integral exists and all that remains is to find a representation for it in this neighborhood. Once the representation is obtained it can be analytically continued to the whole hyperspace $C^4 \times Z_2$.

To evaluate the integral (1), two stock formulas will be used:

(1) Euler representation:

$$z^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} t^{-\alpha-1} e^{-zt} dt , \qquad \mathrm{Re}(z) > 0. \tag{4.4}$$

This representation will be used for exponentiating each and every factor in the integrand in (1).

(2) Gaussian integral in continuous dimensions:

$$\int d^{2\omega} q (q\cdot n)^s \, e^{-\alpha q^2 + 2b\cdot q - \gamma(q\cdot n)^2}$$

$$= \left(\frac{\pi}{\alpha}\right)^{\omega} \left(\frac{b\cdot n}{\alpha+\gamma n^2}\right)^s \left(\frac{\alpha}{\alpha+\gamma n^2}\right)^{1/2} \exp\left\{\left[b^2 - \frac{\gamma(b\cdot n)^2}{\alpha+\gamma n^2}\right]/\alpha\right\} \tag{4.5}$$

This relation, first derived by Capper & Leibbrandt,[4] and being a generalization of (3.8), is needed because of the presence of the factor $(q\cdot n)^{2\nu+s}$ in (2); the integrand is not rotationally invariant (in 2ω-dimension Euclidean space; or not Lorentz invariant in Minkowski space). Eq. (5) is most easily derived in a "cylindrical" coordinate system where the vector n_μ is identified with the z-direction. In this system any vector a_μ can be decomposed as

$$a_\mu = (\bar{a}, a_n), \quad a_n = a\cdot n/(n^2)^{1/2}, \quad \bar{a}^2 = a^2 - a_n^2 \tag{4.6}$$

and a scalar product is given by

$$a\cdot b = \bar{a}\cdot\bar{b} + a_n b_n \tag{4.7}$$

Then

$$\int d^{2\omega} q (q\cdot n)^s \, e^{-\alpha q^2 + 2b\cdot q - \gamma(q\cdot n)^2}$$

$$= (n^2)^{s/2} \int d^{2\omega-1}\bar{q} \, e^{-\alpha\bar{q}^2 + 2\bar{b}\cdot\bar{q}} \int dq_n \, q_n^s \, e^{-[(\alpha+\gamma n^2)q_n^2 - 2b_n q_n]} \tag{4.8}$$

from which (5) is easily verified using standard techniques and (3.8).

Readers interested in the details of how the representation given below is derived should consult the Appendices. The result is[9]

$$S(\omega,\kappa,\mu,\nu,s) = \frac{\pi^{\omega}(p^2)^{\alpha_1}(n^2)^{\nu}(p\cdot n)^{s}\,\Gamma(\nu+s+1/2)}{\Gamma(\beta_1-\alpha_0)\,\Gamma(\beta_1-\alpha_1)\,\Gamma(-\alpha_0-\alpha_1-s)\,\Gamma(-\nu)}$$

$$\cdot\, G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1+\alpha_0,\ 1+\alpha_1,\ 1+\nu;\\ 0,\ \beta_1;\ 1/2-s\end{matrix}\right), \qquad |y| \le 1, \tag{4.9a}$$

$$= \frac{\pi^{\omega}(p^2)^{\alpha_1-\nu}(p\cdot n)^{2\nu+s}\,\Gamma(\nu+s+1/2)}{\Gamma(\beta_1-\alpha_0)\,\Gamma(\beta_1-\alpha_1)\,\Gamma(-\alpha_0-\alpha_1-s)\,\Gamma(-\nu)}\ \cdot$$

$$\cdot\, G^{3,2}_{3,3}\left(\frac{1}{y}\,\middle|\,\begin{matrix}1+\nu,1+\nu-\beta_1;\ \nu+s+1/2\\ 0,\ \nu-\alpha_0,\ \nu-\alpha_1;\end{matrix}\right), \qquad |y| \ge 1 \tag{4.9b}$$

with

$$\alpha_0 = -(\omega+\mu+\nu+s),\quad \alpha_1 = \omega+\kappa+\mu+\nu,\quad \beta_1 = \omega+\kappa+\nu$$

$$y \equiv (p\cdot n)^2/p^2n^2 \tag{4.10}$$

The symbol G is a Meijer G-function.[10] It is a known generalization of the hypergeometric function which can be straightforwardly evaluated. The analytic property of a G-function is most transparent in its contour integral representation

$$G^{mn}_{qp}\left(y\,\middle|\,\begin{matrix}a_1\cdots a_n;\ a_{n+1}\cdots a_q\\ b_1\cdots b_m;\ b_{m+1}\cdots b_p\end{matrix}\right)$$

$$= \frac{1}{2\pi i}\int_L dt\ y^t\,\frac{\left[\prod_{i=1}^{m}\Gamma(b_i-t)\right]\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{i=m+1}^{p}\Gamma(1-b_i+t)\prod_{j=n+1}^{q}\Gamma(a_j-t)} \tag{4.11}$$

where the contour L encloses all poles contained in [ ] but not any others.

Although (9) is derived for the neighbourhood around $\{\omega, \kappa, \mu, \nu, s\} = \{2,-1,-1,-1,1\}$, it can be shown[11] (although we shall not do it here) that the representation S is analytic everywhere in $C^4 \times Z_2$ with, at most, poles possibly when any of the conditions below are met.

$$\alpha_0 = \text{integer} \geq 0 \tag{4.12a}$$

$$\alpha_1 = \text{integer} \geq 0 \tag{4.12b}$$

$$\beta_1 - \nu = \text{integer} \leq 0 \tag{4.12c}$$

or

$$\nu + s + 1/2 = \text{integer} \leq 0 \; . \tag{4.12d}$$

From (9) and (2) we see that (12a,c) are associated with the two types of IR divergences in the integral while (12b) corresponds to the UV divergence. Condition (12d), corresponding to the axial gauge singularity is realized only if ν is a half-integer. But since in the Feynman integral (1) the primal variable N corresponding to ν is always an integer, the representation (9) is free of axial gauge singularities. This result is reminiscent to the regularization of the integral (3.11): the original integral is singular and ill-defined but the regulated representation is completely regular and well-defined.

The regularization of the Feynman integral (1) is summarized in Figure 6.

The power of the generalized representation is demonstrated by considering several special cases:

i) <u>Two-point integrals in covariant gauges ($\nu = s = 0$).</u>

$$\int d^{2\omega}q \frac{1}{[(p-q)^2]^{\alpha}(q^2)^{\beta}} = \frac{\pi^{\omega}(p^2)^{\omega-\alpha-\beta}\,\Gamma(\omega-\alpha)\,\Gamma(\omega-\beta)\,\Gamma(-\omega+\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(2\omega-\alpha-\beta)} \tag{4.13}$$

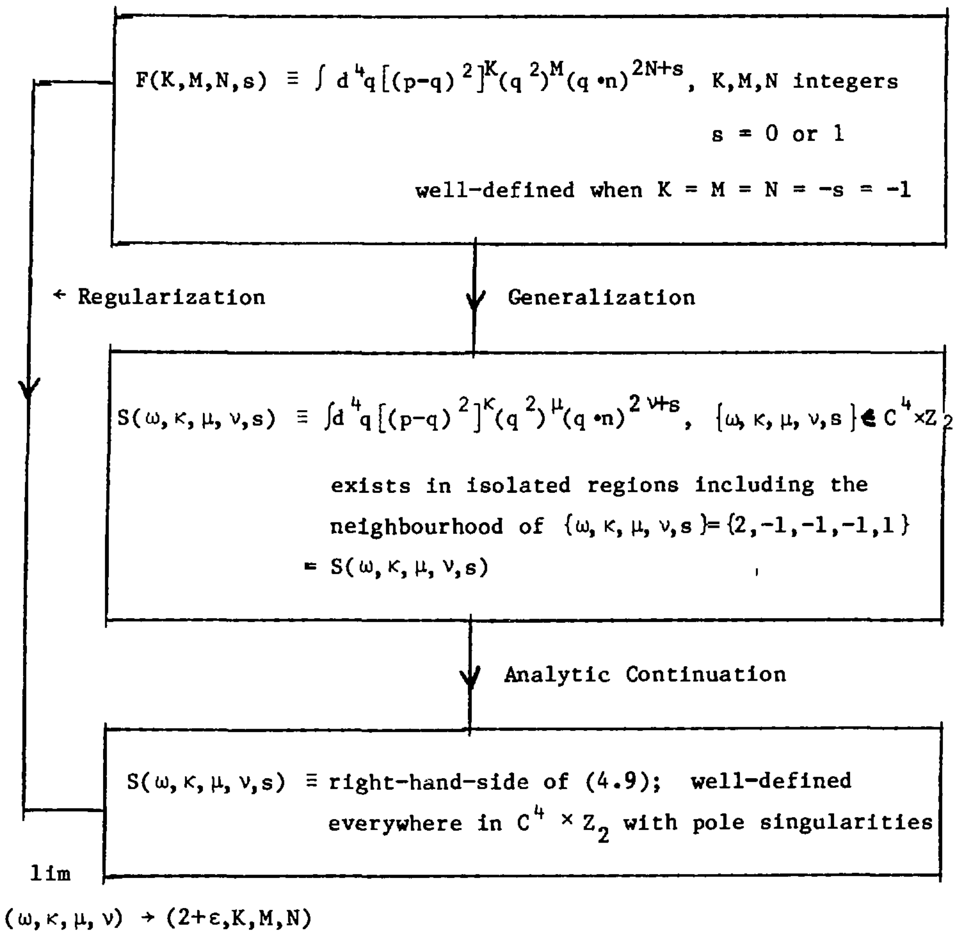

Figure 6

This result reduces to (3.9) when $\alpha=\beta=1$, as it must. However, the representation (13) means all integrals of this type, for any values for α and β, can now be trivially evaluated.

ii) Tadpole integrals. Note that in (13) the representation has a nil value when either α or β is a nonpositive integer. That the integral is symmetric with respect to $\alpha \leftrightarrow \beta$ (the RHS is manifestly so) is a result of the integral being invariant under shifting of the integrated variable. The integral is a tadpole integral when $\alpha=0$. Thus tadpole integrals are just a subclass of a class of nil-valued integrals. The generalized class of tadpole integrals includes

$$S(\omega,K,\mu,\nu,s) = 0\ , \qquad K = \text{integer} \geq 0$$

$$S(\omega,\kappa,M,N,s) = 0\ , \qquad M \text{ and } N = \text{integers} \geq 0 \tag{4.14}$$

Note how the power of analytic continuation has been exploited to derive the result (14). Recall in section 2 we said that tadpole integrals cannot be defined in dim. reg. because a region does not exist in the ω-plane for which such an integral is regular. The more general an. reg. allows one to go beyond the ω-plane to find a region (in $C^4 \times Z_2$) of existence for the generalized integral. After a representation for the generalized integral is found one then returns to the ω-plane by analytic continuation, where one can verify that the integral is indeed nil-valued.

Some readers may wonder how a tadpole can be nil-valued when it may at the same time be UV and IR divergent. The question will be answered in §5 when we learn how to separate the two types of divergences.

iii) Two-point integrals in the light-cone gauge. The light-cone gauge is a special case of the axial gauge defined by the auxiliary constraint

$$n^2 = 0 \tag{4.15}$$

It is a very physical and therefore interesting gauge yet it is notorious for being difficult to regulate. The difficulty originates in the condition (15) which admits a nontrivial solution for the vector

n_μ only in a non-Euclidean space; in Minkowski space with a metric $g_{\mu\nu} = (1,-1,-1,-1)$ one such solution is $n_\mu = (1,0,0,1)$. Since the integral (2) is evaluated in Euclidean space, one must either do the calculation anew for the light-cone gauge, or one may use the already derived result and reach Minkowski space by analytic continuation. As it turns out our result (9) allows for the second option. First we see that (9a) is appropriate for Euclidean space since the inequality

$$(p\cdot n)^2 = p^2 n^2 \cos^2\theta \leq p^2 n^2 \tag{4.16}$$

must always be satisfied in such a space, enforcing the condition $y \equiv (p\cdot n)^2/p^2 n^2 \leq 1$. Conversely, the condition $|y| \geq 1$ for (9b) is never satisfied in Euclidean space, but can be satisfied in Minkowski space. The result (9b) is obtained from (9a) by analytic continuation.[12] In particular, the constraint (15) is reached in the limit

$$1/y \to 0^+ \tag{4.17}$$

in which case (9b) is reduced to the surprisingly simple result

$$\int d^{2\omega}q \, \frac{1}{[(p-q)^2]^\alpha (q^2)^\beta (q\cdot n)^\nu}$$

$$= \frac{\pi^\omega (p^2)^{\omega-\alpha-\beta} (p\cdot n)^{-\nu}\, \Gamma(\omega-\alpha)\,\Gamma(\omega-\beta-\nu)\,\Gamma(-\omega+\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(2\omega-\alpha-\beta-\nu)}\,, \quad (n^2=0) \tag{4.18}$$

for which the similarity to (13) is readily recognized. The light-cone gauge, with its many peculiar properties, is discussed in more detail in §7, where we also give a representation based on the recently devised Mandelstam prescription.[13]

iv) <u>Exponent derivatives</u>. The analyticity of the representation (9) admits the straightforward evaluation of two-point integrals with logarithmic factors in the integrand. Specifically from the relation

$$a^{\varepsilon} = 1 + \varepsilon \ln a + O(\varepsilon^2) \tag{4.19}$$

follows

$$a^{K} \ln^{j} a = \lim_{\kappa \to K} \left(\frac{\partial}{\partial \kappa}\right)^{j} a^{\kappa} \tag{4.20}$$

so that one may derive

$$\int d^{2\omega}q (p-q)^{2K} q^{2M} (q \cdot n)^{2N+s} \ln^{k}[(p-q)^2] \ln^{m}(q^2) \ln^{\ell}(q \cdot n)$$

$$= \lim_{\substack{\kappa \to K \\ \mu \to M \\ \ell \to N}} \left(\frac{\partial}{\partial \kappa}\right)^{k} \left(\frac{\partial}{\partial \mu}\right)^{m} \left(\frac{\partial}{\partial \nu}\right)^{\ell} \int d^{2\omega}q (p-q)^{2\kappa} q^{2\mu} (q \cdot n)^{2\nu+s}$$

$$= \lim_{\substack{\kappa \to K \\ \mu \to M \\ \ell \to N}} \left(\frac{\partial}{\partial \kappa}\right)^{k} \left(\frac{\partial}{\partial \mu}\right)^{m} \left(\frac{\partial}{\partial \nu}\right)^{\ell} S(\omega, \kappa, \mu, \nu, s) \tag{4.21}$$

Thus Feynman integrals with logarithms are just "exponent derivatives"[14] of generalized Feynman integrals without logarithms. Through the representation S, exponent derivatives become normal derivatives.

The potential usefulness of exponent derivatives, although not much explored, is suggested by its occurrence in perturbation field theory. In perturbation theory, Feynman integrals associated with N-loop calculations have integrands with up to (N-1) powers of logarithms, and the evaluated integrals have up to N powers of logarithms. Such terms will appear on the right-hand-side of (21) when $k+m+\ell = N-1$. They arise from taking exponent derivatives of the factor $(p^2)^{\omega+\kappa+\mu+\nu}$ in S. In fact every term, including all proper infinite parts, generated for the two-point function from the multiloop perturbation expansion can be expressed as an exponent derivative of S, while infinite terms which must not appear in the expansion (such as UV infinite terms with a logarithmic dependence on p^2) are never generated in any exponent derivative of S.[11] This raises the speculative but interesting question whether the two-point function can be expressed as the solution of a differential equation having the exponents as variables. Such a solution will in general be a polylog, or a polynomial containing powers of logarithms as well as the usual power terms.

5. Separating UV and IR Singularities

In an. reg. divergences of the two-point integral occurs as poles in C^4 necessarily but not sufficiently when one of the conditions (4.12a, b,c) are met. In terms of the generalized variables these conditions are

$$\omega+\kappa+\mu+\nu = \text{integer} \geq 0 \qquad \text{(UV div.)} \qquad (5.1a)$$

$$\omega+\mu+\nu+s = \text{integer} \leq 0 \qquad \text{(IR div. at } q = o) \qquad (5.1b)$$

$$\omega+\kappa \leq 0 \qquad \text{(IR div. at } q = p) \qquad (5.1c)$$

In Feynman integrals $\omega = 2$ and κ, μ, ν and s are integers. Near these integers we write

$$\begin{aligned} \omega &= 2 + \varepsilon, \\ \kappa &= K + \rho, \\ \mu &= M + \sigma, \\ \nu &= N, \end{aligned} \qquad (5.2)$$

and define

$$\begin{aligned} \varepsilon_1 &\equiv \omega+\kappa+\mu+\nu - (2+K+M+N) = \varepsilon+\rho+\sigma, \\ \varepsilon_0 &\equiv \omega+\mu+\nu+s - (2+M+N+s) = \varepsilon+\sigma, \\ \varepsilon_3 &\equiv \omega+\kappa - (2+K) = \varepsilon+\rho. \end{aligned} \qquad (5.3)$$

When the conditions (1a,b,c) are satisfied, S has the single poles $1/\varepsilon_1$, $1/\varepsilon_0$ and $1/\varepsilon_3$, respectively. Because the three epsilons are distinguishable, the three poles representing the UV and the two kinds of IR singularities can be separately identified. It is now instructive to make a comparison with dim. reg. In that method the exponents K and M (as well as N) are fixed c-numbers, not generalized variables, and the small parameters ρ and σ are by definition identically zero. Then the three ε's are all identically equal to ε

$$\varepsilon_1 = \varepsilon_0 = \varepsilon_3 \equiv \varepsilon, \qquad \text{(dim. reg.)} \tag{5.4}$$

meaning that in dim. reg. it is impossible to identify the origin of poles in the representation.[15]

Later we shall see that Ward-Takahashi identities of Green functions are true only if S is evaluated in the ω-plane (C^1 with κ, μ, ν integer); the identities do not in general hold when S is evaluated in C^4, in particular not in the (ω, κ, μ)-hyperplane C^3. Naturally once we have descended from C^4 to C^1 the ε's cease to be different. However, because the paths of descent for the ε's are all different, each of the ε's can be tagged during the descent so that, even though when in C^1 the three ε's have identical values, their separate identities can be retained.

We have shown that we can distinguish ε_0 from ε_3, if needed. But normally it is unnecessary to separate the two as the following example shows. Consider the integral

$$I = \int d^4q \, \frac{1}{(p-q)^2 q^4}$$

which is IR divergent at q=0 and therefore is expected to have a pole of $O(1/\varepsilon_0)$. But by changing the dummy variable q to p-q we have

$$I = \int d^4q \, \frac{1}{(p-q)^4 q^2}$$

which is now IR divergent at $q = p$ and therefore has a pole of $O(1/\varepsilon_3)$. Since the two integrals are identical we must take a limiting process such that this identity is upheld. This means that in (3) we must have $\rho=\sigma$, so that

$$\varepsilon_0 = \varepsilon_3 = \varepsilon + \rho \ . \tag{5.5}$$

This suggests the following strategy for evaluating two-point integrals

$\boxed{F(K,M,N,s)}$

$\downarrow$ An. reg. - generalize point $(d=4,K,M,N)$ to C^4 to find representation S

$\boxed{S(\omega,\kappa,\mu,\nu,s)}$

$\downarrow$ Evaluation at specific point in C^4 but keeping separate identities for UV and IR singularities by setting $\omega = 2+\varepsilon$, $\kappa = K+\rho$, $\mu = M+\rho$, $\nu = N$

$\boxed{S(2+\varepsilon,K+\rho,M=\rho,N,s)}$

$\downarrow$ Recover gauge invariance by descending from C^4 to ω-plane, taking the limit $\rho\to 0$, $\varepsilon\to$ small. S has poles of $O(1/\varepsilon_1)$ and $O(1/\varepsilon_0)$ representing UV and IR singularities respectively

$\boxed{S(2+\varepsilon,K,M,N,s)}$ (5.6)

In this program (the necessity for taking the last step is explained in §6), the covariant gauge integral at the third stage is given by

$$\int d^4q\ (p-q)^{2K} q^{2M} \rightarrow \pi^{2+\varepsilon}(p^2)^{2+K+M+\varepsilon_1}\Gamma(2+K+\varepsilon_0)\,\Gamma(2+M+\varepsilon_0)\,\cdot$$

$$\cdot\ \Gamma(-2-K-M-\varepsilon_1)/\,\Gamma(-K)\,\Gamma(-M)\,\Gamma(4+K+M+2\,\varepsilon_0) \tag{5.7}$$

The right-hand-side is symmetric under $K \leftrightarrow M$, as it should.

The general axial gauge ($2N+s \neq 0$) integral, for $|y| \leq 1$, is given by

$$\int d^4q(p-q)^{2K}q^{2M}(q\cdot n)^{2N+s} \rightarrow$$

$$\rightarrow \frac{\pi^{2+\varepsilon}(p^2)^{2+K+M+N+\varepsilon_1}\ n^{2N}(p\cdot n)^s\ \Gamma(N+s+1/2)}{\Gamma(-K-\varepsilon_1+\varepsilon_0)}$$

$$\cdot\ \Big\{\frac{\Gamma(2+N+K+\varepsilon_0)\,\Gamma(2+M+N+S+\varepsilon_0)\,\Gamma(-2-K-M-N-\varepsilon_1)}{\Gamma(s+1/2)\,\Gamma(-M-\varepsilon_1+\varepsilon_0)\,\Gamma(4+K+M+2N+s+2\,\varepsilon_0)}$$

$$\cdot\ {}_3F_2\left(\begin{matrix}2+M+N+s+\varepsilon_0, & -2-K-M-N-\varepsilon_1, & -N\\ s+1/2\ , & -1-K-N-\varepsilon_0 & \end{matrix}\ \middle|\ y\right)$$

$$+\ \frac{\Gamma(-2-K-N-\varepsilon_0)\ \Gamma(2+K+\varepsilon_0)}{\Gamma(K+N+s+5/2+\varepsilon_0)\,\Gamma(-N)}\ y^{2+K+s+\varepsilon_0}\ \cdot$$

$$\cdot\ {}_3F_2\left(\begin{matrix}4+K+M+2N+s+2\,\varepsilon_0, & -M-\varepsilon_1+\varepsilon_0, & 2+K+\varepsilon_0\\ K+N+s+5/2+\varepsilon_0, & 3+K+N+\varepsilon_0 & \end{matrix}\ \middle|\ y\right)\Big\} \tag{5.8}$$

where the G-function in (4.9) has been decomposed into a sum of two hypergeometric functions[16] by evaluating the contour integral (4.11).

We now give a few examples.

The formula

$$\Gamma(\varepsilon-\ell) = \frac{(-)^{\ell}\Gamma(1-\varepsilon)\Gamma(1+\varepsilon)}{\varepsilon\Gamma(1+\ell-\varepsilon)}$$

$$= \frac{(-)^{\ell}}{\Gamma(1+\ell)}\left(\frac{1}{\varepsilon}\right)\left[1 + \varepsilon\psi(1+\ell) + O(\varepsilon^2)\right]$$

where the ψ-function satisfies the recurrence relation

$$\psi(z+n) = 1/z + 1/(z+1) + \cdots + 1/(z+n-1) + \psi(z), \qquad (5.10)$$

will be used repeatedly.

(1) F(-1,-1,0,0). This integral, appearing in the evaluation of the one-loop self-energy

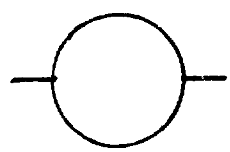

Figure 7

(see Fig. 7) is easily evaluated using (7). The result

$$F(-1,-1,0,0) = \int d^4q(p-q)^{-2}q^{-2}$$

$$\rightarrow \pi^{2+\varepsilon}(p^2)^{\varepsilon_1}\,\Gamma^2(1+\varepsilon_0)\Gamma(-\varepsilon_1)/\Gamma(2+2\varepsilon_0)$$

$$= -\pi^2\left[\frac{1}{\varepsilon_1} + \gamma + \ln\pi + \ln p^2 - 2 + O(\varepsilon)\right] \qquad (5.11)$$

where $\gamma = \psi(1) = 0.577\cdots$ is the Euler-Mascheroni constant, has a UV pole. The three terms in the expression $1/\varepsilon_{1,0} + \gamma + \ln\pi$ always appears in the same combination. In calculations related to field theory the integral usually is multiplied by an extra phase-space factor $(2\pi)^{-4} \to (2\pi)^{-2\omega}$ so that a divergent integral typically has the expansion

$$\frac{1}{(2\pi)^4} \int d^4q \cdots \to \frac{1}{16\pi^2} \left(\frac{1}{\varepsilon_{1,0}} + \gamma - \ln 4\pi + \ln p^2 + \cdots\right) \qquad (5.12)$$

In the renormalization program, infinite parts of the self-energy is absorbed into the wavefunction renormalization, so that the renormalized self-energy is finite. In practice, a Feynman integral for a renormalized quantity is just the integral minus its infinite part. The process of removing the infinite part from a divergent integral is called subtraction. In the minimal subtraction scheme[17] (MS) only the UV pole term $1/\varepsilon_1$ is removed. In the $\overline{\mathrm{MS}}$ scheme[18] the combination $1/\varepsilon_1 + \gamma - \ln 4\pi$ is removed altogether.

For convenience, we define the quantities

$$1/e_{0,1} \equiv 1/\varepsilon_{0,1} + \gamma + \ln \pi \qquad (5.13)$$

The basic motivation for separating the two types of singularities is that in the renormalization program only UV singularities need be subtracted. The removal of IR singularities in field theory is not a completely understood subject. It is generally believed, and proven in (Abelian) quantum electrodynamics,[19] that a process becomes free of IR singularities if all possible ways of emitting soft, massless gauge bosons (photon in QED) are included in the process.

<u>(2) $F(0,-2,0,0)$ ($|y| < 1$)</u>. Returning now to (11), we notice that when $p^2 \to 0$, the expression has a logarithmic singularity. In the special case

when p=0 the integral reduces to a tadpole integral

$$F(0,-2,0,0) = \int d^4q\, q^{-4}$$

$$\to \pi^{2+\varepsilon}(p^2)^{\varepsilon_1}\ \Gamma(2+\varepsilon_0)\Gamma(\varepsilon_0)\Gamma(-\varepsilon_1)/\left[\Gamma(-\varepsilon_1+\varepsilon_0)\Gamma(2-\varepsilon_1+\varepsilon_0)\Gamma(2+2\varepsilon_0)\right]$$

$$= \pi^2\ \left(1/e_0 - 1/e_1\right)\ . \tag{5.14}$$

In the limit $\rho=0$, $\varepsilon_1 = \varepsilon_0 = \varepsilon$, so that the representation is identically zero (it is actually proportional to ρ), giving the usual result for a tadpole integral. The important point is that the integral vanishes as a result of the cancellation between a UV and an IR pole. In other words, if a distinction needs to be made between these two types of poles, then the tadpole integral is not zero. This point is not always realized by those using dim. reg., augmented by the definition that tadpole integrals are nil-valued (see (2.15)), as a regularization tool.

<u>(3) F(-2,-1,1,0)</u>. This integral is encountered when evaluating the one-loop, three-vertex of Fig. 8 in the axial gauge. Because N=1, and $1/\Gamma(-N) = 1/\Gamma(-1) = 0$, the second term in the { } bracket in (8) vanishes. Power counting (consult (1)) tells us that the integral is both UV and IR divergent. Substituting the appropriate integers into (8) yields

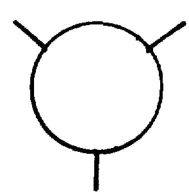

Figure 8

$$F(-2,-1,1,0) \to \frac{\pi^{2+\varepsilon}(p^2)^{\varepsilon} n^2 \, \Gamma(3/2)\,\Gamma(1+\varepsilon_0)\,\Gamma(2+\varepsilon_0)\,\Gamma(-\varepsilon_1)}{\Gamma(1/2)\,\Gamma(2-\varepsilon_1+\varepsilon_0)\,\Gamma(1-\varepsilon_1+\varepsilon_0)\,\Gamma(3+2\varepsilon_0)} \,\cdot$$

$$\cdot \; {}_3F_2\left(\begin{matrix} 2+\varepsilon_0, -\varepsilon_1, -1 \\ 1/2, \; -\varepsilon_0 \end{matrix} \,\middle|\, y \right)$$

$$= \frac{\pi^2 n^2}{2\Gamma(3)} (\pi p^2)^{\varepsilon} \{1 + \varepsilon[\psi(2) - 2\psi(3)]\} \left(\frac{-1}{\varepsilon_1}\right) \left[1 + \frac{(2+\varepsilon)\varepsilon}{(-\varepsilon_0)/2}\right]$$

$$= - \frac{\pi^2 n^2}{16} \left(1/e_1 - 4y/e_0 + 6y - 2\right) \qquad (5.15)$$

The infinite part due to the UV divergence is $-\pi^2 n^2/16e_1$ whereas if UV and IR singularities were not distinguished it would have had an additional multiplicative factor (1-4y).

6. Gauge Invariance and Ward Identities

Because of the gauge invariance of gauge theories, an infinite set of identities, Ward[20] identities for short, exists among various Green functions (n-point functions). A typical Ward identity relates the partial derivative of an (n+1)-point function to a linear combination of n-point functions. Because Ward identities are in general nontrivial equalities, they can be gainfully exploited, among other purposes, to check the consistency of intricate and lengthy computations. For example Ward identities are often used for testing the viability of a regularization method.

The older analytic regularizations mentioned at the beginning of these lectures are known to violate gauge invariance and therefore not to uphold Ward identities in general. In the analytic regularization expounded by Speer,[21] quantum field theory is regulated by modifying propagators, replacing, say (for massless particles) $(p-q)^{-2}$ by $(p-q)^{2\lambda}$. A λ-dependent theory with such modified propagators is free of UV singularities in the complex λ-plane, except for poles at $\lambda = -1$; the theory of interest is recovered in the limit $\lambda \to -1$. However, since the structure of a Lagrangian with a propagator having a continuous exponent is not known, the gauge transformation is not well-defined for theories with $\lambda \neq -1$. Specifically, formal Ward identities - formal because they involve divergent integrals - derived for the real theory are not satisfied in the regularized $\lambda \neq -1$ theories. In this sense Speer's analytic regularization, as well as other analytic methods similar to it, does not preserve gauge invariance.

The crucial difference between our method and the old analytic regularizations lies in two important aspects: (a) the new method is a *technical hybrid of dim. reg. and the generalization of exponents used in* the old analytic method; (b) the generalized dimension and exponents are viewed strictly as a means for regulating divergent integrals, rather than regulating the theory, of which only the four-dimensional one (or whatever integer dimension, as the case may be) is of interest.

Having a generalized dimension is important; we shall see that Ward identities are upheld if and only if (the representations of) Feynman integrals are evaluated in the ω-plane ($\rho = \sigma = 0$, see (5.2)).

For a Yang-Mills theory described by the generating functional $Z[J]$ in the axial gauge (see (1.28)),

$$Z[J] = \int [dA]\, e^{iS_{eff}[A,J]} \tag{6.1}$$

the simplest Ward identities, which are the only ones we shall consider, are derived by considering the variation of $Z[J]$ under the infinitesimal gauge transformation

$$\delta \underline{A}_\mu = \partial_\mu \underline{\Lambda} + (\underline{A}_\mu \times \underline{\Lambda}) \tag{6.2}$$

(the local functions $\Lambda^a(x)$ are infinitesimal),

$$\delta Z[J] = \int [dA] \{[-\partial^x_\mu \delta^{ac} + g f^{abc} A^b_\mu(x)][-\frac{1}{\alpha} n \cdot A^a(x) n_\mu + J^a_\mu(x)]\} \cdot$$

$$\cdot \exp\left(iS_{eff}[A,J]\right) \Lambda^c(x) \ . \tag{6.3}$$

An m-point Ward identity is obtained by taking $(m-1)$ functional derivatives of $\delta Z[J]$ with respect to the source $J^a(x)$ at $(m-1)$ localities, and then evaluating the result at $J = 0$ in the limit $\alpha \to 0$:

$$\lim_{\alpha\to 0}\left\{\prod_{i=1}^{m-1}\frac{\delta}{\delta J^{a_i}(x_i)}\quad(\delta Z[J])\right\}_{J=0}=0 \tag{6.4}$$

The two-point identity, after some manipulation and transformation to momentum space, has the form before the limit $\alpha \to 0$ is taken,

$$p_\lambda \Pi_{\lambda\mu}(p) = i\,\frac{p\cdot n}{\alpha}\,n_\mu \equiv p_\lambda\,\Pi^{(0)}_{\lambda\mu}(p) \tag{6.5}$$

$\Pi_{\lambda\mu}$ is the self-energy to all orders (in g) and $\Pi^{(0)}_{\lambda\mu}$ is the zeroth order, or free, self-energy given in (1.37), from which the second equality sign in (5) is derived. Define

$$\Pi^t_{\lambda\mu} \equiv \Pi_{\lambda\mu} - \Pi^{(0)}_{\lambda\mu} \tag{6.6}$$

to be all the radiative corrections to the self-energy (it will be at least of $O(g^2)$) then from (5) we have the tranversality condition

$$p_\lambda \Pi^t_{\lambda\mu}(p) = 0\ . \tag{6.7}$$

The longitudinal part of $\Pi_{\lambda\mu}$ proportional to $n_\lambda n_\mu/\alpha$ is of no importance. First of all the fact that it appears only in $\Pi^{(0)}_{\lambda\mu}$ and nowhere else means that it is decoupled from the rest of the theory. Secondly it vanishes whenever it is connected to an external gauge field, since the resulting factor $n\cdot\underline{A}$ is zero-valued due to the constraint (1.31). Therefore it need not concern us when (5) appears to diverge in the limit $\alpha\to 0$.

At the one-loop level, $\Pi^t_{\lambda\mu}$ is given by the two diagrams in Fig. 9. Diagram (b) is a tadpole, which we shall take to be zero-valued,

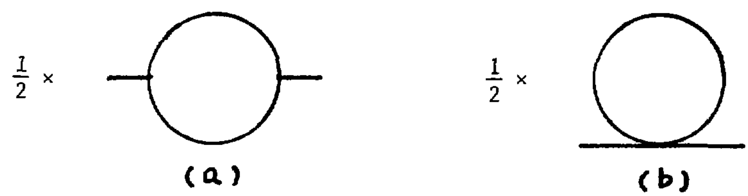

Figure 9

intending for the moment not to distinguish the UV and IR singularities. Diagram (a) is given by

$$\frac{1}{2} \times \left(-\!\!\bigcirc\!\!- \right) = \delta^{ab}\Pi^{(1)}_{\lambda\mu}(p)$$

$$= -\delta^{ab}\,\frac{g^2 C_2}{(2\pi)^4} \int d^4q \left[\Gamma_{\lambda\sigma\tau'}(p,-q,q-p)\,\Delta^{(0)}_{\tau\tau'}(q)\,\Gamma_{\mu\tau\sigma'}(-p,p-q,q)\,\Delta^{(0)}_{\sigma\sigma'}(p-q) \right]_{\alpha=0} \tag{6.8}$$

where the 3-vertex Γ and $\Delta^{(0)}$ are given in §1 and C_2, defined by

$$\delta^{ab} C_2 = f^{acd}\, f^{bcd} \tag{6.9}$$

is the value of the Casmir operator for the adjoint representation of the gauge group; for SU(N), $C_2 = N$. The one-loop self energy, being a rank-2 tensor, involves integrals with tensorial integrands such as

$$\int d^4q\, \frac{q_\mu q_\nu}{(p-q)^2 q^2} \tag{6.10}$$

for which we do not have a generalized representation. The way to evaluate such integrals[22] is to realize that any n-rank tensorial integral can be expressed as a linear combination of products of n-rank tensors constructed from $\delta_{\mu\nu}$, p_μ and n_μ and scalar (or invariant) integrals. For $\Pi^{(1)}_{\mu\nu}$ the expansion (actually true to any order in g) may be written as

$$\Pi^{(1)}_{\lambda\mu}(p) = -i\left[A_1 p^2 \delta_{\lambda\mu} + A_2 p_\lambda p_\mu + A_3 p^2 (p_\lambda n_\mu + p_\mu n_\lambda)/(p\cdot n)\right.$$

$$\left. + A_4 p^4 n_\lambda n_\mu/(p\cdot n)^2\right] \qquad (6.11)$$

where A_i are scalar functions of p^2, n^2 and $p\cdot n$, and are expressible in terms of the four scalar integrals a_i, $i = 1,2,3,4$ defined by

$$-ia_1 = \Pi^{(1)}_{\lambda\mu}\ \delta_{\lambda\mu}/p^2$$

$$-ia_2 = \Pi^{(1)}_{\lambda\mu}\ p_\lambda p_\mu/p^4$$

$$-ia_3 = \Pi^{(1)}_{\lambda\mu}\ p_\lambda n_\mu/(p\cdot n)^2/p^2$$

$$-\ ia_4 = \Pi^{(1)}_{\lambda\mu}\ n_\lambda n_\mu/(p\cdot n)^2 \qquad (6.12)$$

From this example the general procedure for expanding a tensorial integral in terms of scalar integrals becomes clear. Let T_α be an n-rank tensorial integral labelled by α (i.e., $\alpha = \{\lambda\mu\cdots\}$), and $O^{(i)}_\alpha$, $i=1,2,\cdots$ be the complete set of n-rank tensor operators (one of which is $p_\lambda p_\mu\cdots$). Then the scalar functions A_i in the expansion

$$T_\alpha = \sum_i O^{(i)}_\alpha A_i \qquad (6.13)$$

can be expressed in terms of the scalar integrals

$$a_i \equiv O^{(i)}_\alpha T_\alpha . \qquad (\alpha \text{ not summed over}) \qquad (6.14)$$

Substituting (14) into (13) yields

$$a_i = \sum_i O_\alpha^{(i)} \, O_\alpha^{(j)} \, A_j \equiv \sum_j (U^{-1})_{ij} \, A_j \tag{6.15}$$

so that

$$A_i = \sum_j U_{ij} \, a_j \quad . \tag{6.16}$$

In this program, the operations of tensor algebra and the regularization of divergent integrals are completely separated, so that it is not necessary to generalize the algebra originally defined in, say, 4-dimension space to one in 2ω-dimension space. This implies that, among other things,

$$\delta_{\lambda\lambda} = 4 \tag{6.17}$$

rather than $\delta_{\lambda\lambda} = 2\omega$ as in 2ω-dimension space. Thus, for the task at hand, the matrix U for (11) and (12) is a 4×4 matrix, with

$$(U^{-1})_{11} = \delta_{\lambda\mu}\delta_{\lambda\mu} = 4 \qquad (\text{not } 2\omega) \tag{6.18}$$

The scalar integrals a_i can now be reduced to a form suitable for representation. For example, suppose

$$T_\alpha \rightarrow T_{\mu\nu} = \int d^4q \, \frac{q_\mu n_\nu}{(p-q)^2 q^2} \tag{6.19}$$

and

$$O_\alpha^{(i)} \rightarrow O_{\mu\nu}^{(i)} = p_\mu p_\nu \tag{6.20}$$

Then

$$a_1 = (p\cdot n) \int d^4q \frac{p\cdot q}{(p-q)^2 q^2} = \frac{1}{2} \int d^4q \frac{[p^2+q^2-(p-q)^2]}{(p-q)^2 q^2}$$

$$= (p\cdot n) \left[\frac{p^2}{2} \int d^4q \frac{1}{(p-q)^2 q^2} + \frac{1}{2} \int d^4q \frac{1}{(p-q)^2} - \frac{1}{2} \int d^4q \frac{1}{q^2}\right]$$

$$= \frac{(p\cdot n)p^2}{2} \int d^4q \frac{1}{(p-q)^2 q^2} \qquad (6.21)$$

The last two terms on the second-to-last line cancel, being equivalent by the shift operation.

In evaluating (11) and (12) we will also encounter integrals such as

$$I = \int d^4q\ K(p,q) \frac{1}{(q\cdot n)[(p-q)\cdot n]} \qquad (6.22)$$

The standard technique to be used here is partial fraction:[22]

$$I = \int d^4q K(p,q) \frac{1}{p\cdot n} \left[\frac{1}{q\cdot n} + \frac{1}{(p-q)\cdot n}\right]$$

$$= \frac{1}{p\cdot n} \int d^4q\ [K(p,q)/(q\cdot n) + K(p,p-q)/(q\cdot n)] \qquad (6.23)$$

where the second kernel in [ ] comes from changing variable $q \rightarrow p-q$. This technique can be applied repeatedly if necessary.

We are now ready to explain why it is necessary to take the last step in (5.6), i.e., sending $\rho=\sigma=0$ in (5.2), if Ward identities are to hold. The reason is that many of the manipulations used to reduce the integrals to forms suitable for generalized representation are only applicable to primal integrals - integrals with only integer exponents. It follows that Ward identities are true only for expressions involving primal integrals - integrals with all κ, μ, and ν being integers. This is why Speer's analytic regularization does not uphold Ward identities, whereas our method does, provided the representation for the integrals are evaluated for integer exponents as charted out in (5.6).

In the axial gauge the actual computation reducing the A_i's in (11) to linear combinations of primal integrals is rather lengthy[31], and the resulting expressions are too long to be given here. However, computations involved in (12), (16) (i.e. computing the reciprocal of U^{-1}), and manipulations analogous to (21) and (23) can all be carried out with the aid of algebraic computer programs such as SCHOONSCHIP, REDUCE II or MACSYMA. Let us simply take for granted that the A_i's have been thus calculated. Then, substituting (11) into (7) we have

$$ip_\lambda \, \Pi^{(1)}_{\lambda\mu}(p) = (A_1+A_2+A_3)\, p^2 p_\mu + (A_3+A_4)p^4 n_\mu/(p\cdot n) = 0 \qquad (6.24)$$

implying that the A_i's must satisfy

$$A_1 + A_2 = -A_3 = A_4 \; . \qquad (6.25)$$

Remarkably, the A_i's we have computed reduce to linear combinations of primal integrals that satisfy (25) identically, provided we let all tadpole integrals be zero. Recall that tadpoles vanish exactly only if the associated exponents are integers; so we see once again the necessity of setting $\sigma = \rho = 0$ (see (5.2)).

Because the Ward identities (25) are satisfied at the integral level - i.e., before the integral has been evaluated - it is clear that they are still satisfied when the <u>scalar integrals</u> are generalized to 2ω-dimension, regardless of the value of ω. This strengthens our conviction that algebraic manipulation and regularization are two operations that can, and should, be separated.

With the A_i's satisfying the Ward identities (25), (11) can be rewritten as

$$\Pi^{(1)}_{\lambda\mu}(p) = -i\left(\Pi_0\, P_{\lambda\mu} + \Pi_1\, N_{\mu\nu}\right),\quad \Pi_0 = A_1,\quad \Pi_1 = -A_3, \tag{6.26}$$

$$P_{\lambda\mu} \equiv p^2\delta_{\lambda\mu} - p_\lambda p_\mu$$

$$N_{\lambda\mu} \equiv \left[p_\lambda - p^2 n_\lambda/(p\cdot n)\right]\left[p_\mu - p^2 n_\mu/(p\cdot n)\right] \tag{6.27}$$

It is clear that both of the tensors $P_{\lambda\mu}$ and $N_{\lambda\mu}$ are perpendicular to p_λ so (26) is guaranteed to satisfy the transversality condition (7).

When all the integrals are evaluated we find

$$\Pi_0 = \frac{g^2C_2}{32\pi^2}\,\frac{1}{1-\zeta}\,\Big[\frac{22}{3e}(1-\zeta) - \ln\left(\frac{4}{\zeta}\right)(8 - 6\zeta + \zeta^2) - \frac{62}{9} + \frac{44\zeta}{9} + 2\zeta^2$$
$$+ \left(\frac{8}{\zeta} - 8 + 2\zeta - \frac{\zeta^2}{2}\right) Z\Big]$$

$$\Pi_1 = \frac{g^2C_2}{32\pi^2}\,\frac{1}{1-\zeta}\,\Big[-\frac{10}{3} + 2\zeta + \ln\left(\frac{4}{\zeta}\right)\left(7 - \zeta - \frac{9}{1-\zeta}\right)$$
$$-\frac{1}{2}\left(\frac{16}{\zeta} - 5 + \zeta - \frac{9}{1-\zeta}\right)Z\Big] \tag{6.28}$$

where $1/e$ is defined as in (5.13),

$$Z = 2\sum_{\ell=0}^{\infty}\frac{(1)_\ell\, y^\ell}{(3/2)_\ell}\left[\ln y - \psi(1+\ ) + \psi(3/2+\ )\right],\quad |y| \le 1 \tag{6.29}$$

is proportional to the finite integral $S(2,-1,-1,-1,1) = \pi^2 y\, Z/(p\cdot n)$, $\zeta = 1/y$. Note that only Π_0, the radiative correction to the zeroth order self-energy has an infinite part. We do not at this stage know whether this is a UV or an IR infinite term however.

To find out the origin of the infinite term, we use the limiting process described in the last section (see (5.6)) to delineate the two types of singularities. In this calculation we must also take into account the contribution from diagram (b) of Fig. 8, since tadpoles do not vanish when UV and IR poles are counted separately. The result of this calculation is as follows: (a) the relations (25) are no longer manifestly satisfied at the integral level, but are satisfied when the primal integrals are evaluated; this probably implies that there exist identities among primal integrals of which we are not yet aware. (b) The values for Π_0 and Π_1 are identical to those in (28), except that the pole term $1/\varepsilon$ in (29) must now be replaced by $1/\varepsilon_1$; the self-energy has only a UV infinite part, but is IR finite.

Thus, following the usual renormalization procedure, we can remove this infinite part by adding to the original Lagrangian a counter-term corresponding to the kinetic energy,

$$\Delta\mathcal{L} = \frac{g^2 C_2}{16\pi^2}\,\frac{11}{3\varepsilon}\left(\partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu\right)^2 \tag{6.30}$$

We now briefly discuss the verification of the three-point Ward identity,

$$ip_\lambda\,\Gamma^{abc}_{\lambda\mu\nu}(p,q,r) = g\,f^{abc}\left[\Pi^t_{\mu\nu}(q) - \Pi^t_{\mu\nu}(r)\right] \tag{6.31}$$

derived from first taking two functional derivatives of $\delta Z[J]$ and then taking the Fourier transformation. Computation of the general one-loop three-vertex function involves the evaluation of three-point integrals, for which we do not have a generalized representation. We therefore examine only the special case, with $q = -p$, $r = 0$. The Ward identity of interest is

$$ip_\lambda \, \Gamma^{(1)}_{\lambda\mu\nu}(p,-p,o) = g \, \Pi^{(1)}_{\mu\nu}(p) \tag{6.32}$$

where a factor of f^{abc} has been removed from both sides. The three-vertex is represented by the diagrams in Fig. 10.

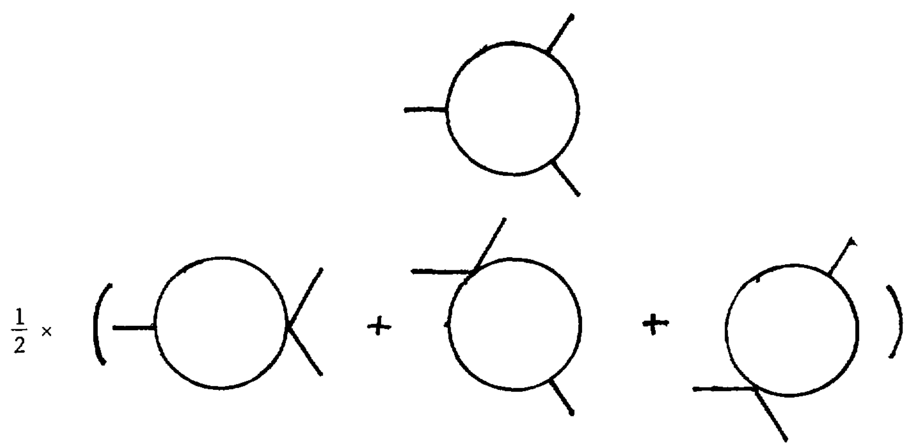

Figure 10

It can be expanded as described in (13) to (16) so that again only scalar two-point integrals need be evaluated. Again we find that: (a) if UV and IR singularities are not separated, then the Ward identity is manifestly satisfied at the primal integral level; (b) if they are separated, then the identity is satisfied when the integrals are evaluated (through the generalized representation), and $\Gamma^{(1)}$ has only UV infinite parts; all IR singularities having cancelled among themselves. The result[31] is

$$\Gamma^{(1)}_{\lambda\mu\nu}(p,-p,0) = \frac{g^3C_2}{16\pi^2}\,\frac{11}{3e_1}\left(2\delta_{\lambda\mu}p_\nu - \delta_{\mu\nu}p_\lambda - \delta_{\lambda\nu}p_\mu\right) + \text{finite parts} \qquad (6.33)$$

implying for the general case

$$\Gamma^{(1)}_{\lambda\mu\nu}(p\ q\ r) = \frac{g^2C_2}{16\pi^2}\,\frac{11}{3e_1}\,\Gamma^{(0)}_{\lambda\mu\nu}(p,q,r) + \text{finite parts} \qquad (6.34)$$

In the MS scheme[17], the wavefunction renormalization constant Z_3 and the vertex renormalization constant Z_1 are defined respectively via

$$\left(\Pi^t_{\mu\nu} - \Pi^{(0)t}_{\mu\nu}\right)_{\text{inf}} = (Z_3-1)\ \Pi^{(0)t}_{\mu\nu}\ , \qquad (6.35a)$$

$$\left(\Gamma_{\lambda\mu\nu} - \Gamma^{(0)}_{\lambda\mu\nu}\right)_{\text{inf}} = (Z_1-1)\ \Gamma^{(0)}_{\lambda\mu\nu} \qquad (6.35b)$$

From (28) and (34), we see that at the one-loop level

$$Z_1 = Z_3 = 1 + \frac{g^2C_2}{16\pi^2}\,\frac{11}{3\varepsilon_1} \qquad (6.36)$$

The equality of Z_1 and Z_3 is special to axial gauges, but not generally true in nonaxial gauges.

To conclude this section, we have demonstrated that when properly applied, analytic regularization preserves gauge invariance. The key point is that after using the generalized representation $S(\omega,\kappa,\mu,\nu,s)$ to evaluate Feynman integrals $F(K,M,N,s)$ we must take the limit $\kappa \to K$, $\mu \to M$. We also showed that we can use the limiting process to separate UV singularities from IR singularities without violating Ward identities. The method used here to isolate algebra from the analysis for regularization also strongly suggests that even though the regularization employs dim. reg., it may not be necessary to generalize the algebra to 2ω-dimension space. This conjecture is certainly true for the limited cases studied here, but a more extensive investigation is needed before it can be taken as generally valid. In view of the recent controversy on the question whether supersymmetric theories can be quantized because a regularization obeying all supersymmetries may not exist, the task of searching for a regularization that works independently of algebra becomes more urgent.

7. The Light-Cone Gauge

7.1 Principal-Value Prescription

The light-cone gauge[23] is a special axial gauge constrained by the additional condition

$$n^2 = 0 \tag{7.1}$$

Since only a nil-vector has zero-norm in Euclidean space the constraint (1) can be nontrivial only in a nonEuclidean space, such as the Minkowski space. Conventionally, integrals in the axial gauge have been derived (mostly using the principal-value prescription) in the Euclidean space, in which (1) cannot be met, so that the integrals had to be derived anew for the light-cone gauge, and this had led to the belief that the light-cone gauge is not a special case of the axial gauge.

In our analytic method, which gives results equivalent to those derived from the principal-value prescription, the representation for the generalized integrals, although derived in Euclidean space, is sufficiently analytic to admit continuation back to Minkowski case, so that the representation for light-cone gauge integrals actually is a special case of the representation for axial gauge integrals.

In recent years the light-cone gauge, in spite of being especially singular, has gained increased popularity because it is (a) ghost-free; (b) at least superficially simple; and (c) physical. It is ghost-free because it is an axial gauge. It is superficially simple because, compared to (1.40), the propagator simplfies to

$$\lim_{\alpha \to 0} \Delta^{(0)ab}_{\lambda\mu} = i \frac{\delta^{ab}}{p^2} \left(\delta_{\lambda\mu} - \frac{p_\lambda n_\mu + p_\mu n_\lambda}{p \cdot n}\right) \tag{7.2}$$

As well, we have already shown in §4 that two-point integrals in the light-cone gauge are enormously more simple than the general axial gauge integrals (see (4.18)).

Another feature adding to the attractiveness of the light-cone gauge is that it allows one to work explicitly with only two of the four components of the gauge field. Let us first choose n_μ to be (in Minkowski space with metric (1,-1,-1,-1))

$$n_\mu = (1,0,0,1)/\sqrt{2} \tag{7.3}$$

Now any vector a^μ can be decomposed into the two components

$$a^\pm = (a^0 \pm a^3)/\sqrt{2} \tag{7.4a}$$

and the two-component vector that lives on the xy-plane

$$\hat{a} = (0, a^1, a^2, 0) . \tag{7.4b}$$

Similarly a contravector have components

$$a_\pm = (a_0 \pm a_3)/\sqrt{2} = a^\mp , \tag{7.5a}$$

$$a_i = -a^i , \qquad i = 1,2 . \tag{7.5b}$$

The scalar product is

$$\begin{aligned} a\cdot b = a_\mu b^\mu &= a_+b^+ + a_-b^- + \sum_{i=1}^{2} a_i\, b^i \\ &= a^+b^- + a^-b^+ - \hat{a}\cdot\hat{b} \end{aligned} \tag{7.6}$$

In particular

$$a \cdot n = a^{+} \; . \tag{7.7}$$

The axial gauge condition therefore reads

$$n \cdot A^{a} = A^{a+} = 0 \; . \tag{7.8}$$

The component A^{a^-} can also be eliminated from the theory by making use of the equation of motion

$$\frac{\partial \mathcal{L}}{\partial A^{a-}} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial A^{a-}_{\mu}} = -\partial^{\mu}\partial_{+}A^{a}_{\mu} - gf^{abc}\, A^{b}_{\mu}\partial_{+}A^{c\mu} = 0 \tag{7.10}$$

yielding

$$A^{a-} = \frac{\partial^{i}}{\partial^{+}} A^{ai} + gf^{abc} \frac{1}{(\partial^{+})^{2}} \; A^{bi}\partial^{+} A^{ci} \quad . \tag{7.11}$$

This means that the theory, which had a four-component gauge field A^{a}_{μ} to start with, can in the light-cone gauge be reduced to a theory involving explicitly only two of the components.

The reduced theory[26] has a particularly simple free boson propagator

$$\Delta^{(0)ab}_{ij} = i \, \frac{\delta^{ab}\delta_{ij}}{p^{2}} \qquad\qquad i,j = 1,2 \tag{7.12}$$

but has somewhat more complicated 3 and 4 vertices:

$${}_{3}\Gamma^{(0)abc}_{ijk}(p,q,r) \equiv gf^{abc}\; \Gamma^{(0)}_{ijk}(p,q,r)$$

$$= gf^{abc} \; \{\delta_{ij}[(p-q)_{k} - (p-q)^{+} \frac{r_{k}}{r^{+}}]$$

$$+ \delta_{jk}[(q-r)_{i} - (q-r)^{+} \frac{p_{i}}{p^{+}}] + \delta_{ki} \; [(r-p)_{j} - (r-p)^{+} \frac{q_{j}}{q^{+}}]\} \tag{7.13}$$

$$ {}_4\Gamma^{(0)abcd}_{ijk\ell}(p,q,r,s) = -ig^2 \; \{ f^{abe} f^{cde} [2\,\delta_{ij}\,\delta_{k\ell} \frac{q^+ s^+}{(p+q)^+ (r+s)^+} + \delta_{ik}\delta_{j\ell}] + \text{permutations} \} \tag{7.14} $$

One may choose to use for computation either the Feynman rules (12-14) or (7) and (1.38,39). The final result should be equivalent. Because we want to compare our light-cone gauge calculation with the results in §6, we choose to use the latter set of Feynman rules, in which *case the $A^{a\pm}$ fields are not eliminated.*

We first describe how the result (4.18) is derived by analytic continuation. We shall consider $n^2 = 0$ as a limit of $n^2 \to 0^+$. In this limit $y = (p\cdot n)^2/p^2 n^2 \to \infty^+$, so our starting point is (4.9b),

$$ S(\omega, \kappa, \mu, \nu, s) = \cdots \times G^{3,2}_{3,3}\,(1/y \mid \cdots), \qquad |y| \geq 1. $$

Near $1/y = 0$, schematically[16]

$$ G^{3,2}_{3,3}(1/y \mid \cdots) = y^0 \;{}_3F_2(\cdots \mid 1/y) + y^{\omega+\kappa+\mu}\;{}_3F_2(\cdots \mid 1/y) + y^{-\omega-\mu-\nu-s}\;{}_3F_2(\cdots \mid 1/y) \tag{7.15} $$

where the (three different sets of) variables for the ${}_3F_2$ functions have been suppressed. The RHS is well-defined only in regions of $C^4 \times Z_2$ space where $\omega+\kappa+\mu < 0$ and $-\omega-\mu-2\nu-s < 0$. Two regions in which these conditions as well as the conditions for which the original integral exists (see (4.12)) are the neighborhoods of $\{\omega, \kappa, \mu, \nu, s\} = \{2,-1,-2,0,1\}$ and $\{2,-1,-3,1,1\}$. In any such region, in the limit $1/y \to 0^+$,

$$y^0 \; {}_3F_2(\cdots 1/y) \to 1;$$

the second and third terms on the RHS of (15) vanish, so that

$$S(\omega,\kappa,\mu,\nu,\sigma) \to L(\omega,\kappa,\mu,\bar{\nu} = 2\nu{+}s)$$

$$\equiv \frac{\pi^{\omega}(p^2)^{\omega+\kappa+\mu}(p\cdot n)^{\bar{\nu}}\;\Gamma(\omega+\kappa)\,\Gamma(\omega+\mu+\bar{\nu})\,\Gamma(-\omega-\kappa-\mu)}{\Gamma(-\kappa)\,\Gamma(-\mu)\,\Gamma(2\omega+\kappa+\mu+\bar{\nu})}\;, \qquad (7.16)$$

which is equivalent to (4.18). Now, the RHS of (16) is well-defined in all of $C^4 \times Z_2$ with at most pole singularities, so by the principle of analytic continuation it is a representation for S in the whole space, when $1/y = 0^+$, i.e., when $n^2 = 0$.

We now discuss the Ward identities in the light-cone gauge, first without attempting to separate UV and IR singularities. We find that again both the two and three-point identities are manifestly satisfied at the primal integral level, provided tadpole integrals are discarded (their representation (16) are nil-valued). The one-loop self-energy[22], which has the form (6.26), is simple enough to be given here,

$$\Pi_0 = \frac{g^2 C_2}{32\pi^2}\,(-4L_2 - 6L_0 + 8L_{-1})$$

$$\Pi_1 = \frac{g^2 C_2}{32\pi^2}\,(12L_2 + 4L_0 - 8L_{-1}) \qquad (7.17)$$

with

$$L_{\bar{\nu}} \equiv \pi^{-2}(p\cdot n)^{-\bar{\nu}}\;L(\omega,-1,-1,\bar{\nu}) \qquad (7.18)$$

The full expression[22] for $\Gamma^{(1)}_{\lambda\mu\nu}(p,-p,0)$ is still too lengthy to be given here but its contraction satisfies the three-point identity (6.32) and therefore also has the form (6.26). So far nothing sets the light-cone gauge apart from the other axial gauges except its relative simplicity. Its peculiarity is exposed only when the integrals in (17) are scrutinized.

When we evaluate the integrals in (17) using (16), we find

$$L_0 = -1/e + 2$$

$$L_2 = -\ 1/3e + 13/18$$

$$L_{-1} = -\left\{\frac{1}{\varepsilon}\left(\frac{1}{e} + \ln p^2\right) + \frac{1}{2}\left[(\gamma + \ln p^2)^2 - \pi^2/6\right]\right\} \qquad (7.19)$$

where $1/e$ is defined in (5.13). This result is unusual in two important aspects:

(i) The integral L_{-1} contains a double pole of $O(1/\varepsilon^2)$ and a logarithmic single pole of $O(\ln p^2/\varepsilon)$. The latter is particularly bothersome because it cannot be removed by a local counterterm.

(ii) The function Π_1 has an infinite part. This means that a counterterm in addition to (6.30) and having the form $\partial_\mu A^\mu \partial_\nu A^\nu$ is needed for renormalization; n_μ-dependent terms are not needed because $n\cdot A = 0$.

The first point casts the renormalizability of light-cone gauge in doubt, insofar as the usual method of using counterterms is concerned. Whether a viable renormalization scheme can be found for our regularization of the light-cone gauge is a question that has not yet been answered.

The strange result for the light-cone gauge can be understood by examining more closely the analytic continuation used to derive (16) from

(15). For any finite value of $1/y$, the integral may be finite even when more than one of the three terms on the RHS of (15) have infinite parts. An example is the integral L_{-1}, for which the first term in (15) is responsible for the poles given in (19), including the double poles. For finite $1/y$ all of these poles are however cancelled by poles contained in the second and third terms in (15); the integral is finite in axial gauges with $n^2 \neq 0$, see (6.29). In the limit $1/y \to 0^+$, the second and third terms are discarded (see discussion following (16)) and the cancellation effect is lost. The key point here is that a representation with drastically different analytic properties is obtained if the limit $1/y \to 0^+$ is taken before all others. As an independent check of the correctness of (16) (in the particular limiting process under discussion) an identical result can be derived[24] by setting $n^2=0$ at the outset and taking steps analogous to those described in Appendix B. Integrals evaluated from (16) also agree, as expected, with those computed using the principal-value prescription according to (2.18).

Some of the peculiar properties of (16) are:

(i) The conditions for having pole singularities are different from those of the general case:

(a) UV sing. when $\omega+\kappa+\mu \geq 0$

(b) IR sing. ($q=0$) when $\omega+\mu+\nu \leq 0$

(c) IR sing. ($q=p$) when $\omega+\kappa \leq 0$. (7.20)

Only (c) is the same as before (see (5.1). It follows that in this regularization power counting is lost, explaining why the integral corresponding to L_{-1} in the axial gauge is finite (see (6.29), but has single and double poles in the light-cone gauge;

(ii) UV and IR singularities are indistinguishable. This shows up when one attempts to separate UV and IR singularities in the three point Ward identity; the UV-infinite and IR-infinite terms do not separately satisfy the identity (they do in axial gauge with $n^2 \neq 0$);

(iii) "Unrenormalizable" singularities of order $O(1/\varepsilon^2)$ and $O(\ell n\, p^2/\varepsilon)$ appear in one-loop calculations, as noticed earlier.

9.2 The Mandelstam Prescription

In view of the undesirable properties of the regularization (16), there have been recent attempts to find new regularizations that may have better properties. One such is proposed by Leibbrandt[25] where

$$\frac{1}{q \cdot n} \stackrel{\text{def}}{=} \frac{-(\vec{q}\cdot\vec{n} + i\,|\vec{n}|q_4)}{(\vec{q}\cdot\vec{n})^2 + \vec{n}^2\, q_4^2} \tag{7.21}$$

(The scalar product $q \cdot n = q_4 n_4 + \vec{q}\cdot\vec{n}$ in Euclidean space, and the light-cone condition $n^2=0$ is satisfied by setting $n_4 = \pm i|\vec{n}|$). The prescription retains power counting, has only single poles and has been shown to satisfy the two-point Ward identity at the one-loop level, but is Lorentz-noninvariant.

Another Lorentz-noninvariant prescription, devised by Mandelstam,[13] uses the replacement

$$\frac{1}{q^+} \rightarrow \frac{1}{[q^+]} \stackrel{\text{def}}{=} \lim_{\eta\to 0^+} \frac{1}{q^+ + iq^-\eta} \tag{7.22}$$

where $q^{\pm}$ is defined as in (4a) and η is a small c-number to be set to zero after integration. Mandelstam used this prescription to prove the finiteness of the N=4 supersymmetric Yang-Mills theory. Despite appearances, the two prescriptions (21) and (22) have been shown to be equivalent.[32]

Recently Capper et al.[26] showed that the light-cone gauge integral L(2,-1,-1,-1) in the Mandelstam prescription is finite, in sharp contrast to the result (19) obtained in the principal-value prescription. We are therefore motivated to find a representation for the generalized class of two-point light-cone gauge integrals based on Mandelstam's prescription. As explained in Appendix C, in this prescription it is necessary to evaluate the integral in Minkowski space.

We define the generalized integral as

$$M(\omega,\kappa,\mu,\nu) \equiv \lim_{\eta\to 0^+} \int_{\text{Minkowski}} d^{2\omega}q\,[(p-q)^2 + i\eta)^{\kappa}(q^2+i\eta)^{\mu}\,(q^+ + i\eta q^-)^{\nu} \qquad (7.23)$$

for which we find the representation (for derivation see Appendix C)[32]

$$M = M_0\, z^{-\nu}\, G^{2,3}_{3,3}\left(z \,\middle|\, \begin{matrix} 1-\omega-\mu, 1+\omega+\kappa+\mu+\nu, 1+\nu; \\ 0,\ \omega+\kappa+\nu;\ \nu \end{matrix}\right), \quad |z| \le 1\ ,$$

$$= M_0\, G^{3,2}_{3,3}\left(\frac{1}{z} \,\middle|\, \begin{matrix} 1+\nu, 1-\omega-\kappa; 1 \\ 0, \omega+\mu+\nu, -\omega-\kappa-\mu; \end{matrix}\right), \quad |z| \ge 1$$

$$M_0 \equiv \frac{i(\pi e^{-i\pi})^{\omega}(p^2)^{\omega+\kappa+\mu}(p^+)^{\nu}}{\Gamma(-\kappa)\,\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(2\omega+\kappa+\mu+\nu)}$$

$$z \equiv 2p^+p^-/p^2 \qquad (7.24)$$

where $p^{\pm}$ are the light-cone variables in (4a). This result has some similarity to the one given in (4.9) but the two are obviously not identical.

In particular the extra phase factor of $ie^{-i\pi\omega}$ in (24) comes from the fact that (23) is defined as an integral in Minkowski space. Aside from this phase factor, the two sets of results are expected to be identical when ν is a non-negative integer N. Indeed one can show that

$$M(\omega,\kappa,\mu,N\geq 0) = \frac{i(\pi e^{-i\pi})^{\omega}(p^2)^{\omega+\kappa+\mu}(p^+)^{\nu}\Gamma(\omega+\kappa)\Gamma(\omega+\mu+N)\Gamma(-\omega-\mu-\kappa)}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(2\omega+\kappa+\mu+N)} \tag{7.25}$$

which is identical to $L(\omega,\kappa,\mu,\bar{\nu}=N)$ of (16) to within a phase factor. This implies that the Mandelstam prescription still does not obey normal power counting for $\nu = N \geq 0$, since UV divergence is determined by the abnormal condition $\omega+\mu+\nu$ = integer ≥ 0.

We now examine the especially interesting integral with $\kappa=\mu=\nu=-1$, which is the one (and only one) integral in the principal-value prescription to become a regular but nonterminating series in the axial gauge ((6.29)) and to have a double pole and other peculiar properties discussed earlier in the light-cone gauge ((17)). In the Mandelstam prescription, from (24), we find it to be finite

$$\begin{aligned} M(2,-1,-1,-1) &= \frac{i\pi^2}{p^+}\, z\left[{}_3F_2\left({1,1,1 \atop 2,2}\middle| z\right) - \ln z\ {}_2F_1\left({1,1 \atop 2}\middle| z\right)\right], \qquad |z| \leq 1, \\ &= \frac{i\pi^2}{p^+}\left[\left\{\frac{\pi^2}{3} - \frac{1}{2}\ln^2 z + z^{-1}\left[{}_3F_2\left({1,1,1 \atop 2,2}\middle| z\right) - \ln z\ {}_2F_1\left({1,1 \atop 2}\middle| z\right)\right]\right\}\right], \quad |z| \geq 1, \end{aligned} \tag{7.26}$$

in accordance with power counting. The one-loop correction to the self-energy can now be read off from (17), (19) and (26), remembering that L_2 and L_0 are the same in the two prescriptions, and that L_{-1} is to be replaced by (26). We find for the infinite parts

$$(\Pi_0)_{inf} = \frac{g^2 C_2}{16\pi^2} \frac{11}{3\varepsilon} ,$$

$$(\Pi_1)_{inf} = \frac{g^2 C_2}{16\pi^2} \frac{4}{\varepsilon} . \tag{7.27}$$

Of interest is that $(\Pi_0)_{inf}$ is identical to its counterpart in the axial gauge in the limit $y \to \infty$ (see (6.28)). Still present is the infinite part in Π_1, necessitating an extra counterterm for renormalization.

We briefly summarize some other results:

(i) The three-point Ward identity is separately satisfied for the UV-divergent, IR-divergent and finite parts;

(ii) As in the axial gauge, infinite parts in both $\Pi_{\mu\nu}$ and $\Gamma_{\lambda\mu\nu}$ are all of UV origin; all IR singularities cancel among themselves;

(iii) The infinite parts of $\Gamma_{\lambda\mu\nu}$ are not the same as in (6.34);

(iv) The renormalization constants Z_1 and Z_3 are not equal, contrary to (6.36).

This suggests that in the Mandelstam prescription, the light-cone gauge may still require an unusual renormalization program; our calculation shows that the theory in this gauge is probably not multiplicatively renormalizable.

Noted added: *Very recent and preliminary results[32] suggest that the light-cone gauge in the Mandelstam prescription requires only the normal renormalization program, and that $Z_1 = Z_3$, provided one works in the two-component theory described in Eqs. (8-14).*

8. RENORMALIZATION, THE β-FUNCTION, AND ASYMPTOTIC FREEDOM

Although many quantities in gauge theories are gauge-dependent, the physics described in such theories must be gauge-independent. One of the gauge-independent properties in nonAbelian theories is asymptotic freedom.[27] The coupling constant g for the interaction in an asymptotically free theory becomes vanishingly small in the limit when the momentum λ characterizing a physical event is increasingly greater than a certain fixed momentum scale λ_0, rendering the theory interaction-free. This property can be expressed as

$$\lim_{\lambda/\lambda_0 \to \infty} g^2(\lambda) \to 0 \, . \tag{8.1}$$

The fixed momentum scale λ_0 can only be determined experimentally.

Asymptotic freedom is a result of radiative renormalization effects symptomatic of all field theories. The calculations we have already done in the last two sections for the one-loop corrections for the self-energy $\Pi_{\mu\nu}$ and the three-vertex $\Gamma_{\lambda\mu\nu}$ are sufficient for the discussion of this topic. We shall show that although the self-energy and the three-vertex are not renormalized the same way in axial gauges as they are in covariant gauges, the two classes of gauges yield identical quantitative results for asymptotic freedom.

For theories that are multiplicatively renormalizable (*for the light-cone gauge see note at the end of §7*) the behaviour of g as a function of λ is characterized by its logarithmic derivative with respect to λ, known as the β-function

$$\beta(g) \equiv \lambda \frac{\partial g(\lambda)}{\partial \lambda} \, . \tag{8.2}$$

the λ-dependence of g comes via the wavefunction renormalization constant Z_3 and the vertex renormalization constant Z_1 which are related to g and the bare coupling constant g_o by

$$g_o = gZ_1Z_3^{-3/2} \equiv gZ_g \tag{8.3}$$

where Z_3 and Z_1 defined by

$$\begin{aligned} \Pi^t_{\mu\nu} &= Z_3\ \Pi^{(0)t}_{\mu\nu} \\ \Gamma_{\lambda\mu\nu} &= Z_1\ \Gamma^{(0)}_{\lambda\mu\nu} \end{aligned} \tag{8.4}$$

(the superscript t denotes the transverese part of $\Pi_{\mu\nu}$, see §6), embody radiative corrections which we have calculated to lowest order in g^2 in §§6 and 7. From (6.28,34) these renormalization constants are the same in the axial gauge ($n^2 \neq 0$).

$$Z_1 = Z_3 = 1 + \frac{g^2C_2}{16\pi^2}\ \frac{11}{3}\ \Big[\frac{1}{\varepsilon} + \ln(p^2/\lambda^2) +$$

$$+ (\lambda\text{-independent regular parts})\Big]\ . \tag{8.5}$$

We have for the first time explicitly displayed the dependence of the logarithmic term on the scale momentum λ. For a massless theory where there are no other momentum to serve as a dimensionful scale, it is clear, from dimensional arguments, that λ must enter in this way. Our calculations have shown that there are no other terms in Z_1 that are dimensionful - the variable $y = (p \cdot n)^2/p^2n^2$ in the axial gauge is dimensionless. For massive theories there may be terms such as m^2/λ^2, but their logarithmic derivatives always vanish in the asymptotic limit $\lambda \to \infty$.

In §5 we pointed out that, in radiative corrections, the logarithmic and pole terms always occur in the same linear combination as in (5) (see also (5.12)). The origin of this correlation lies in the expansion

$$(p^2/\lambda^2)^{\varepsilon}\,[\tfrac{1}{\varepsilon} + O(\varepsilon^{o})] = [1 + \varepsilon\ln(p^2/\lambda^2) + O(\varepsilon^2)][\tfrac{1}{\varepsilon} + O(\varepsilon^{o})]$$

$$= \frac{1}{\varepsilon} + \ln(p^2/\lambda^2) + O(\varepsilon^{o}) \qquad (8.6)$$

This implies that for the purpose of calculating the β-function to lowest order in g^2, only the infinite parts of the renormalization constants are needed, since

$$\lambda \frac{\partial Z_i}{\partial\lambda} = -\,2\,\frac{\partial Z_i}{\partial(1/\varepsilon)}\,. \qquad (8.7)$$

Let us define the coefficients, b, b_1 and b_3 via

$$(Z_g)_{\text{infinite}} = -\,(\tfrac{1}{2}\,bg^2)\,\frac{1}{\varepsilon} \qquad (8.8)$$

$$(Z_{1,3})_{\text{infinite}} = (b_{1,3}\,g^2)\,\frac{1}{\varepsilon} \qquad (8.9)$$

then from (3) and (5), for the axial gauge

$$b_{YM} = (b_1)_{YM} = (b_3)_{YM} = \frac{11C_2}{48\pi^2} + O(g^2)\;, \qquad \text{(axial gauge)} \qquad (8.10)$$

where the subscript YM denotes contribution from Yang-Mills fields only. The significance of this relation, arising from the equivalence of the renormalization constants Z_1 and Z_3, is that in the axial gauge the renormalization of the self-energy alone determines the β-function. This relation does not in general hold in nonaxial gauges. Indeed, to lowest order for covariant gauges[28]

$$(b_1)_{YM} = \frac{C_2}{32\pi^2}\left(\frac{17}{6} - \frac{3\alpha}{2}\right) \quad ,$$

$$(b_3)_{YM} = \frac{C_2}{32\pi^2}\left(\frac{13}{3} - \alpha\right) \quad , \qquad \text{(covariant gauges)} \tag{8.11}$$

where α is the gauge-fixing parameter (see (1.24)); the inequivalence of b_1 and b_3 results from the existence of ghosts. However, from (3), (8) and (11),

$$b_{YM} = \frac{11C_2}{48\pi^2} + O(g^2) \quad , \qquad \text{(gauge independent)} \tag{8.12}$$

showing that the coefficient b is a gauge-independent quantity.

We now proceed to demonstrate asymptotic freedom. Because g_o is independent of λ, we obtain from (2), (3), (7) and (12), the gauge-independent β-function

$$\beta(g) = 2g\; Z_g^{-1}\left[\partial Z_g/\partial(1/\varepsilon)\right]$$

$$= -\, bg^3 + O(g^5) \tag{8.13}$$

Because $C_2 \geq 0$ and therefore $b_{YM} > 0$ to lowest order, the negative sign in (13) implies asymptotic freedom. For if there exists a momentum λ_o for which g^2 is sufficiently small for the leading term in (13) to dominate when $\lambda >> \lambda_o$, then the solution for $g^2(\lambda)$ in the asymptotic region $(\lambda/\lambda_o >> 1)$ is

$$g^2(\lambda) = \left[b\,\ell n(\lambda^2/\lambda_o^2)\right]^{-1} \tag{8.14}$$

Asymptotic freedom as prescribed by (1) then follows. We emphasize that the constant momentum scale λ_0 is not calculable in the theory; its empirical value[29] is 250 ± 150 MeV/c.

We now briefly discuss what roles fermions, which we have ignored so far, play in asymptotic freedom and why an Abelian theory such as quantum electrodynamics is not asymptotic-free. The fermionic contributions to the renormalization constants can also be calculated from the diagrams in Figs. 8 and 9, but with all internal (gauge field) lines replaced by fermion lines. The gauge independent result is[28]

$$b_{fermion} = (b_1)_{fermion} = (b_3)_{fermion} = -\frac{N_f}{24\pi^2} \tag{8.15}$$

where N_f is the number of fermion species. Significant is the contrast between the signs in (15) and (11): whereas $b_{YM} \geq 0$, $b_{fermion} < 0$. The total value for b is thus

$$b = b_{YM} + b_{fermion} = \frac{1}{24\pi^2}\left(\frac{11C_2}{2} - N_f\right) \tag{8.16}$$

This means that a theory with N_f fermions is asymptotically free only if

$$N_f \leq \frac{11C_2}{2} \tag{8.17}$$

A nonAbelian theory such as quantum chromodynamics, with gauge group SU(3) and $C_2 = 3$, is therefore asymptotically free if

$$N_f \leq 16 \qquad (\text{for } SU(3)_{color}) \tag{8.18}$$

On the other hand, any Abelian theory must not be asymptotically free, since $C_2 = 0$ (see (6.9); the structure constant is zero for an Abelian group) and $N_f > 0$.

APPENDIX A - INTEGRATION IN EUCLIDEAN AND MINKOWSKI SPACES

For invariant integrals, where all quantities involved in the integrand are scalar products, the only thing that matters in deciding whether the integration space should be Euclidean or Minkowskian is the possible range of values for the norm of a vector. For this determines how factors appearing in a Feynman integral can be exponentiated.

Exponentiation is generally necessary before integration in a generalized continuous dimensional space can be carried out. The only formula needed for this task is probably also the most useful formula for evaluating Feynman integrals, namely the Euler's formula

$$z^{\mu} = \frac{1}{\Gamma(-\mu)} \int_{o}^{\infty} dt\ t^{-\mu-1}\ e^{-zt}\ , \qquad \mathrm{Re}(z) > 0,\ \mathrm{Re}(\mu) < 0\ . \tag{A.1}$$

The constraint $\mathrm{Re}(z) > 0$ is of special interest to us.

In Euclidean space the norm of a vector not being nil is always positive definite

$$p^2 = p_{\mu} p_{\mu} > 0\ , \tag{A.2}$$

so one can use (A.1) simply by replacing the z there by p^2 (or p^2+m^2 for massive integrals).

In Minkowski space with metric (1,-1,-1,-1), because the norm of a vector is indefinite,

$$p^2 = p_{\mu} p^{\mu} = p_0^2 - \vec{p}^{\,2}\ , \tag{A.3}$$

one must use (A.1) in a more round-about way:

$$(p^2)^\mu \overset{\text{def}}{=} \lim_{\eta \to 0^+} \left[p^2 + i\eta\right]^\mu$$

$$= \lim_{\eta \to 0^+} i^\mu \left[-i\; p^2 + \eta\right]^\mu$$

$$= \lim_{\eta \to 0^+} \frac{\mu}{\Gamma(-\mu)} \int_0^\infty dt\; t^{-\mu-1}\; e^{ip^2 t - \eta t} \quad , \qquad \text{(Minkowski)}. \qquad (A.4)$$

One can see how the small $\eta > 0$ term is needed to satisfy the first constraint for (A.1); the limit $\eta \to 0^+$ is to be taken after integration.

APPENDIX B - DERIVATION OF (4.9)

The integral

$$S \equiv S(\omega,\kappa,\mu,\nu,s) = \int d^{2\omega}q\,[(p-q)^2]^{\kappa}(q^2)^{\mu}[(q\cdot n)^2]^{\nu}(q\cdot n)^{s} \tag{B.1}$$

is evaluated in Euclidean space. Use (A.1) to exponentiate the three factors $(p-q)^2$, q^2 and $(q\cdot n)^2$ separately, and then use (4.8) to do the q-integration to obtain

$$\begin{aligned} S &= S_0 \int_0^\infty dt \int_0^\infty du \int_0^\infty dv\; t^{\mu-1}\, u^{-\nu-1} v^{-\kappa-1+s} (t+v)^{\omega-1/2} \\ &\quad \times (t+u+v)^{s+1/2} \exp\Big[\Big(\frac{v}{t+v}\Big)\Big(t - \frac{vy}{t+u+v}\Big)\Big] , \\ S_0 &\equiv \pi^{\omega}(p\cdot n)^{s}(n^2)^{\nu}(p^2)^{\alpha_1}/[\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)] , \\ y &\equiv (p\cdot n)^2/p^2n^2 , \\ \alpha_1 &\equiv \omega+\kappa+\mu+\nu . \end{aligned} \tag{B.2}$$

Now transform the variables

$$\begin{aligned} u &= \lambda\tau, \\ t &= \lambda(1-\tau)\xi, \\ v &= \lambda(1-\tau)(1-\xi) . \end{aligned} \tag{B.3}$$

The Jacobian is $\partial(u,t,v)/\partial(\lambda,\tau,\xi) = \lambda^2$ and the ranges for integration are $(0,1)$ for τ and ξ and $(0,\infty)$ for λ. The λ-integration is easily done - again using (A.1) - to obtain

$$S = S_o\ \Gamma(-\alpha_1) \int_0^1 d\xi\ \xi^{\beta_1 -1} (1-\xi)^{-\alpha_0 -1} \times$$

$$\times \int_0^1 d\tau\ \tau^{-\nu-1} (1-\tau)^{\nu+s-1/2} [1 + y\tau(1-\xi)/\xi]^{\alpha_1} \tag{B.4}$$

where $\alpha_0 = -\omega-\mu-\nu-s$ and $\beta_1 = \omega+\kappa+\nu$ as well as α_1 are the same parameters given in (4.10). The τ-integration can be identified as a hypergeometric function ${}_3F_2$ (see Luke,[10] Sec. 3.6.(1); p.57), which can be expressed as a G-function $G^{1,2}_{2,2}$ (Luke, Sec. 5.2.(14); p.147) so that after a further transformation

$$v = (1-\xi)/\xi\ , \tag{B.5}$$

we have

$$S = S_0\ \Gamma(\nu+s+1/2) \int_o^\infty dv\ v^{-\alpha_0 -1}\ (1+v)^{-\beta_1+\alpha_0}\ G^{1,2}_{2,2}\left(yv\,\middle|\,\begin{matrix}1+\alpha_1,\ 1+\nu; \\ 0;\ 1/2-s\end{matrix}\right) \tag{B.6}$$

The integral is known, yielding (Luke, Sec. 5.6(18); p.165)

$$S = \frac{S_0\Gamma(\nu+s+1/2)}{\Gamma(\beta_1-\alpha_0)}\ G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1+\alpha_0, 1+\alpha_1, 1+\nu; \\ 0,\ \beta_1;\ 1/2-s\end{matrix}\right) \tag{B.7}$$

which is the result for $|y| \leq 1$ given in (4.9a). The result (4.9b) for $|y| \geq 1$ is obtained by analytic continuation (Luke, Sec. 5.4.(3); p.150).

The reference used in (5.8) and (7.15) for expanding the G-function in terms of hypergeometric functions is (Luke, Sec. 5.2.(7); p.145).

APPENDIX C - DERIVATION OF (7.24)

Mandelstam's prescription for the light-cone gauge is devised explicitly for integration in Minkowski space, so according to (7.22) and (A.4), we define

$$M(\omega,\kappa,\mu,\nu) \equiv \lim_{\eta\to 0^+} \int d^{2\omega}q\,[(p-q)^2 + i\eta]^{\kappa}\,(q^2+i\eta)^{\mu}\,(q^+ + i\eta q^-)^{\nu} \qquad (c.1)$$

the evaluation of which always depends on a generic integral over the whole xy-plane

$$I \equiv \lim_{\eta\to 0^+} \int dxdy(x+i\eta y)^{\nu} e^{2i(ax+by+cxy)}\,, \quad \mathrm{Re}(c) > 0\,. \qquad (C.2)$$

In order to exponentiate the factor $(x+i\eta y)^{\nu}$ with the aid of (A.1) we integrate over the upper and lower-half y-plane separately and obtain

$$I = \frac{\pi}{c\Gamma(-\nu)} \lim_{\eta\to 0^+} \int_0^\infty dt\; t^{-\nu-1} e^{-\eta t} \int_0^\infty dy \;\times$$

$$\times\; \left[i^{\nu} e^{2iby}\, \delta\!\left(y + \frac{t}{2c} + \frac{a}{c}\right) + (-i)^{\nu} e^{-2iby}\, \delta\!\left(y + \frac{t}{2c} - \frac{a}{c}\right)\right] \qquad (C.3)$$

where the δ-functions are from the x-integrations. The result for (C.3) is easily shown to be proportional to a confluent hypergeometric function ${}_1F_1(-\nu;1-\nu;2iab/c)$ (Luke, Sec. 3.1.(18);p.40). However, for our purposes we write

$$I = \frac{\pi}{c\Gamma(-\nu)} (2ia)^{-\nu} e^{-2iab/c} \lim_{\eta\to 0^+} \int_0^1 dt\; t^{-\nu-1}\, e^{2iabt/c-\eta t}\,. \qquad (C.4)$$

Now use this result to derive the equivalent of (4.8) for Mandelstam's prescription in Minkowski space

$$J(c,p_\mu) \equiv \lim_{\eta\to 0^+} \int d^{2\omega}q(q^+ + i\eta q^-)^\nu e^{i(cq^2-2p\cdot q)}$$

$$= \cdots \int d^{2(\omega-1)}\hat{q} \int dq^+ dq^- \cdots$$

$$= \frac{i}{\Gamma(-\nu)} \left(\frac{-i\pi}{c}\right)^\omega (-2ip^-)^{-\nu} e^{-ip^2c} \int_0^1 dt\ t^{-\nu-1} e^{2ip^+p^-t/c} \tag{C.5}$$

The light-cone decomposition of vectors was explained in (7.4-6). We now return to (C.1) and exponentiate the first two factors according to (A.4) to find

$$M = \frac{i^{\kappa+\mu}}{\Gamma(-\mu)\Gamma(-\kappa)} \lim_{\eta\to 0^+} \int_0^\infty dr \int_0^\infty ds\ r^{-\kappa-1} s^{-\mu-1} e^{-\eta(r+s)} e^{ip^2r} J(r+s, rp_\mu). \tag{C.6}$$

After substituting (6) into (5), changing variables

$$\begin{aligned} r &= \lambda(1-\xi) \\ s &= \lambda\xi \end{aligned} \tag{c.7}$$

and integrating over λ (from 0 to ∞), we obtain

$$M = \frac{i(\pi e^{-i\pi})^\omega (p^2)^{\alpha_1} (2p^-)^{-\nu} \Gamma(-\alpha_1)}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)} \int_0^1 d\xi\ \xi^{\beta_1-1} (1-\xi)^{-\alpha_0'-1} \times$$

$$\times \int_0^1 dt\ t^{-\nu-1} [1 + zt(1-\xi)/\xi]^{\alpha_1} \tag{C.8}$$

where $\alpha_0' = -\omega-\mu$ and $z = 2p^+p^-/p^2$. This integral is identical in form to (B.4), and can be evaluated following the same steps as those given in (B.5-7), yielding (7.24) for $|z| \leq 1$. The result for $|z| \geq 1$ is again obtained by analytic continuation.

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