TANGLES, LINKS AND TWISTED QUANTUM GROUPS

H.C. Lee

Theoretical Physics Branch, Chalk River Nuclear Laboratories
Atomic Energy of Canada Limited, Research Company
Chalk River, Ontario, Canada K0J 1J0
and
Dept. of Applied Mathematics, University of Western Ontario
London, Ontario, Canada N6A 5B9

1. Introduction

In this paper we report three main results: (a) An algebraic-geometric construction of universal link invariants in quantum groups based on Reidemeister's theorem; (b) A similar construction of universal tangle invariants -- diagrammatically a tangle is a link diagram with one external edge cut -- and a proof showing that, with a certain restriction on the quantum group, the set of all tangle invariants forms a subset of the centre of the quantum group; (c) A demonstration that a classical simple Lie (and Kac-Moody) algebra can be at least twice deformed to give a "twisted" quantum group with an associated twisted Hopf structure -- the classical Alexander-Conway link Polynomial is the simplest link invariant constructed from the twisted quantum group of $\mathfrak{sl}(2)$. These results are used to derive a number of theorems on the quantum group invariants of tangles, knots and links and their representations. They are also used to give insight to our understanding of the relation between quantum groups

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and conformal and topological field theories. It appears that, effectively, only the maximum Abelian subalgebra of the Hopf structure of the quantum group acts in these theories. It is shown that a tangle to a link in knot theory is what a "Wilson tangle" is to a Wilson line in the Chern-Simons-Witten topological field theory. The tangle theorem derived in (b) asserts that the set of all Wilson tangles forms a U(1) group, which is a natural holonomy group of the CSW theory.

In the last few years evidence has been accumulating to show that link invariants and quantum groups[1,2,3] provide an underlying structure common to a large and diverse array of topics in mathematics and low-dimension physics including braid group representations,[3-6] knot and link theory,[3-9] fractional statistics,[10-13] exactly solvable[14] statistical and lattice models,[2,2a,4,6,15] conformal field theory in two dimensions[16] and quantum theory[17] and topological field theory[18] in three dimensions. Most constructions of link invariants in quantum groups have been based on representation theory. In this scheme, a representation of the universal $\mathscr{R}$-matrix of the quantum group is used to generate a representation for Artin's braid group,[19] from which a link invariant is then constructed using Markov's theorem.[20] Recently Lawrence,[9] still utilizing the braid group and Markov's theorem, constructed a link invariant that is a universal (*i.e.*, representation independent) invariant of the quantum group. Our construction is based on Reidemeister's theorem[21]; it makes reference to neither the braid group nor Markov's theorem. While our approach is very geometric, we also make full use of the algebraic properties of the quantum group.

Our construction is similar to the state-model construction of link invariants (not universal) by Kauffman[8], also based on Reidemeister's theorem (but not in the context of a quantum group). It is closest in spirit to Witten's construction in the context of a topologically invariant field theory in three dimensions -- in our construction one may view elements of the Hopf algebra as particles moving in a three-dimensional manifold acted on by the topological Chern-Simons action.

Another aspect that sets the present approach apart from previous ones is the identification of *three* elements in the quantum group as fundamental for the construction of a universal link invariant. For terminology, by quantum group we shall mean the Hopf algebra, or the universal enveloping algebra $\mathscr{A}$ of the q-analogue $\mathscr{g}'$ of the the Lie algebra $\mathscr{g}$. The three fundamental elements are the well-known universal matrix $\mathscr{R} \in \mathscr{g}' \otimes \mathscr{g}'$, the universal element $\mathscr{h} \in \mathscr{g}'$, and the central element $\lambda \in$ centre of $\mathscr{g}'$. A careful study of the properties of these elements and their relation to the link invariant yields the second and perhaps most important result of this paper: the set of all tangle invariants in a quasitriangular quantum group whose $\mathscr{h}$ is not unipotent in $\mathscr{g}'$/centre forms a subset of the centre of the quantum group. We believe all quantum groups of simple Lie algebras belong to the above restricted type. The relation between the tangle invariant $\mathscr{V}$ and the associated link invariant P is simple: since $\mathscr{V}$ is in the centre of $\mathscr{g}'$, $\mathscr{V} = \check{\mathscr{V}} e_0$, where e_0 is the identy element in $\mathscr{g}'$; then $P = \check{\mathscr{V}}\,\mathrm{Tr}(\mathscr{h})$ to within a normalization (given explicitly in the text), where Tr maps $\mathscr{g}'$ to $\mathbb{C}$ and is invariant under cyclic permutations of its argument. In matrix representation, Tr is just the trace of the matrix. One may view the universal tangle invariants as an infinite set of Casimir operators of the quantum group, and think of P as their $\mathscr{h}$-weighted traces. There is however an important difference in the relations between a Casimir operator and its trace in a Lie algebra and between $\mathscr{V}$ and P in a quantum group. In a Lie algebra, the ratio of the eigenvalues of a Casimir operator and its trace is just the dimension of the representation. In a quantum group the ratio, proportional to the eigenvalue of $\mathrm{Tr}(\mathscr{h})$, is a nontrivial functional of the representation and may also be a function of the deformation