

Tangle invariants and centre of the quantum group

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Abstract. An ambient isotopy $\mathcal{V}$ for (n, n) -tangles $T^{(n)}$, $\mathcal{V} : T^{(n)} \rightarrow \mathcal{A}^{\otimes n}$ is constructed on the quasitriangular Hopf algebra $\mathcal{A}$ of a quantum group. $\mathcal{V}[T^{(0)}]$ is an invariant for links. A theorem stating that the set $\{\mathcal{V}[T^{(1)}] \mid \text{all } T^{(1)}\} \subset \text{centre of } \mathcal{A}$ is proved. This implies that $\mathcal{V}[T^{(1)}]$ is also an invariant of links. In particular, if $\pi \in \text{End}(V_{\mathcal{A}})$ is an irreducible representation of $\mathcal{A}$, then for all $(1, 1)$ -tangles $T^{(1)}$ whose closure is L , a link polynomial $Q_{\pi}[L] \in C[[q]]$ is given by the relation $\pi(\mathcal{V}[T^{(1)}]) = Q_{\pi}[L]\pi(1)$, and redundantly by $\pi(\mathcal{V}[L]) = Q_{\pi}[L]\text{Trace}_{\pi}(h)$, where h is a universal element of $\mathcal{A}$ and q denotes a generic deformation parameter of the quantum group. The isotopy is extended to cabled tangles using the coproduct in $\mathcal{A}$. Two examples, when π is the fundamental 2×2 representation of $sl(2; C)_q$ and when it is that of $(gl(2; C)/U(1))_{q,s}$, are discussed. The $Q_{\pi}[L]$ in these cases are respectively the Jones and Alexander-Conway polynomials.

1. Quantum group and Hopf algebra

We understand a quantum group in the sense of Drinfeld [1]. Specifically it is a quantized universal enveloping algebra of a Lie algebra g , denoted by $\mathcal{U}_q(g)$, or simply $(g)_q$ in this paper. It has a Hopf algebra structure composed of the set $\{\mathcal{A}, m, \Delta, u, \epsilon, S\} = \{\text{bialgebra}, \text{product}, \text{coproduct}, \text{unit}, \text{counit}, \text{antipode}\}$ over the field $V_{\mathcal{A}}$. Often we refer to the Hopf algebra as $\mathcal{A}$. The maps m, Δ, u and ϵ are homomorphisms; S is an antihomomorphism. The following relations are especially important for our purpose:

$$(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a), \quad (1.1)$$

$$m(id \otimes S)\Delta(a) = m(S \otimes id)\Delta(a) = u\epsilon(a), \quad \forall a \in \mathcal{A}. \quad (1.2)$$

2. Quasitriangular $\mathcal{R}$ -matrix

It is assumed that $\mathcal{A}$ is quasitriangular, namely that there exists an invertible universal $\mathcal{R}$ -matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, whose inverse is $\mathcal{R}^{-1} = (S \otimes id)\mathcal{R} = (id \otimes S^{-1})\mathcal{R}$, that satisfies the relations

$$\mathcal{T} \circ \Delta(a)\mathcal{R} = \mathcal{R}\Delta(a), \quad \forall a \in \mathcal{A}, \quad (2.1)$$

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (2.2)$$

where if $\mathcal{R} = \sum_i a_i \otimes b_i$, then, e.g., $\mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i$, and so on. The braid relation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \quad (2.3)$$

is a consequence of the above two relations.

From its property of quantum doubling [1], $\mathcal{A}$ has Hopf subalgebras $\mathcal{A}^+$ and $\mathcal{A}^-$ which are dual to each other, so that the (opposite) coproduct on $\mathcal{A}^+$ ($\mathcal{A}^-$) induces the product on $\mathcal{A}^-$ ($\mathcal{A}^+$). This means that if $\{e_\sigma\}$ is a basis for $\mathcal{A}^-$, then [1,2] there exists a basis $\{e^\sigma\}$ for $\mathcal{A}^+$ dual to $\{e_\sigma\}$, and

$$\mathcal{R} = \sum_{\sigma} e_{\sigma} \otimes e^{\sigma}. \quad (2.4)$$

For quantum groups that are quantized universal enveloping algebras of simple Lie algebras $\mathcal{A}$ are finitely generated infinite algebras so that the summation over σ in (2.4) is infinite. This makes the derivation of certain properties of a quasitriangular Hopf algebra problematic. (For instance (2.1) and (2.2) express the same relation if the sum were finite.) In the following we take another approach to the description of $\mathcal{R}$ so that the summation in (2.4) may be replaced by the limit of a finite summation.

Consider $\mathcal{A}^{\pm}$ as linear spaces respectively spanned by $\{e_{\sigma}\}$ and $\{e^{\sigma}\}$. We want to construct a series of finite subspaces

$$S_{N_1}^{\pm} \subset S_{N_2}^{\pm} \dots \subset S_{\infty}^{\pm} = \mathcal{A}^{\pm}, \quad N_1 < N_2 \dots \infty \quad (2.5)$$

each having the property

$$\Delta(S_N^{\pm}) \subset S_N^{\pm} \otimes S_N^{\pm}, \quad (2.6)$$

This can be achieved because coproduct is homogeneous in the lowering (and raising) generators. In the case of $sl(2; C)_q$ with generators [2] $\{k, k^{-1}, X^{\pm}\}$ and coproduct $\Delta(X^{\pm}) = X^{\pm} \otimes k + k^{-1} \otimes X^{\pm}$ we let $S_N^{\pm}$ be spanned by the smallest set containing $(X^{\pm})^N$. Then the ideals $\mathcal{I}_N^{\pm}$ generated by $(X^{\pm})^{N+1}$ are $\not\subset S_N^{\pm}$. For example S_1^+ is spanned by $\{k^{-1}, k, X^{\pm}\}$, $S_3^+ \supset S_1^+$ by $\{k^{-3}, k^{-1}, k, k^3, k^{-2}X^{\pm}, X^{\pm}, k^2X^{\pm}, k^{-1}(X^{\pm})^2, k(X^{\pm})^2, (X^{\pm})^3\}$, etc. In general,

$$\forall S_N^{\pm}, \quad \exists N' > N \text{ such that } S_N^{\pm} \subset S_{N'}^{\pm}. \quad (2.7)$$

The generalization to quantum groups with higher ranks is straightforward.

from above and exit from below. Closing a strand means joining the two ends of a strand in an area external to the interior strip. Closing a tangle means closing all strands of a tangle. The result is the *closure* of the tangle. In this work we restrict our consideration to those tangles that can be closed without generating any new crossings. In what follows, all Reidemeister moves [5] are restricted to within the interior strip.

Consider an operation, called splicing, under which a positive (negative) crossing is replaced by an *uncrossing*, represented by a plus (minus) sign and two noncrossing strands, as shown below:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \sim \left. \begin{array}{c} | \\ | \end{array} \right) + \left(\begin{array}{c} | \\ | \end{array} \right. , \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \sim \left. \begin{array}{c} | \\ | \end{array} \right) - \left(\begin{array}{c} | \\ | \end{array} \right. \quad (4.1)$$

The result of splicing all the crossings in a $T^{(n)}$ is the union of a set of n noncrossing strands, m unknots, called Seifert circles, and uncrossings located at the vertices of $T^{(n)}$. In a spliced tangle, called a *splice*, the relative positions of uncrossings along any strand or Seifert circle may not be altered. An *untangle* is a tangle that has no crossings. The following lemma is self evident:

Lemma 4.2. *Splicing is a bijection between tangles and splices.*

Consider a Seifert circle with s uncrossings around its perimeter which divide its circumference into s sections. If $s > 1$ the Seifert circle may be deformed such that no two uncrossings are allowed to be on the same latitude so that the sections point generally either upward or downward; at least one section will point up and at least one down. If $s = 1$ we consider the sole section to be pointing up and call the Seifert circle trivial. A tangle is said to be trivial if its splice contains at most only trivial Seifert circles; otherwise it is nontrivial. A trivial tangle is equivalent to a braid (by the Reidemeister move I, see below). The following has been proven elsewhere [22]:

Lemma 4.3. *It is possible to deform, without generating any additional crossing, a tangle to one such that each of the Seifert circles in the splice contains one and only one section pointing up.*

5. Reidemeister moves

We construct a homomorphism $\mathcal{V}$ between the complete set of (n, n) -tangles and $\mathcal{A}^{\otimes n}$ and define the Reidemeister moves under $\mathcal{V}$. It is then shown that under $\mathcal{V}$ the tangles are invariant under the Reidemeister moves.

The homomorphism $\mathcal{V}$ is a natural one: each strand in the tangle is mapped to an element in an $\mathcal{A}$ factor in $\mathcal{A}^{\otimes n}$. The actions of $\mathcal{V}$ on a local part of a tangle are defined as

$$\downarrow \rightarrow 1 \in \mathcal{A}, \quad (5.1a)$$

$$\left. \begin{array}{l} \downarrow \\ \downarrow \end{array} \right\} \rightarrow 1 \otimes 1 \in \mathcal{A} \otimes \mathcal{A}, \quad (5.1b)$$

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow e_\sigma \otimes e^\sigma = \mathcal{R} \in \mathcal{A} \otimes \mathcal{A}, \quad (5.1c)$$

$$\left. \downarrow \right\} \rightarrow h \in \mathcal{A}, \quad (5.1e)$$

(in a counterclockwise circle)

$$\left\{ \downarrow \right. \rightarrow h^{-1} \in \mathcal{A}. \quad (5.1f)$$

This explains the need for (5.1e and f). The general rules for the insertion of h and/or h^{-1} factors on a tangle T is:

- Splice T and call the result P . P is the union of a set of noncrossing strands, uncrossings and Seifert circles.
- On each counter-clockwise (clockwise) Seifert circle in P , insert a factor of h (h^{-1}) on the upward-pointing section (each Seifert circle has one and only one such). Call the result P' .
- Generate from P' a unique tangle T' by inverse splicing (by Lemma 4.1). T' is the same as T except that it has factors of h and/or h^{-1} inserted on all the sections having the wrong directions.

An example suffices to explain these rules.



Henceforth it will be understood that by a tangle T we mean T' .

It follows that (3.5) and (3.6), respectively, induce the equivalence relations

$$\begin{array}{c} \text{Diagram 1} \end{array} \sim \begin{array}{c} \text{Diagram 2} \end{array} \sim \begin{array}{c} \text{Diagram 3} \end{array} \quad (5.4b)$$

$$\begin{array}{c} \text{Diagram 4} \end{array} \sim \begin{array}{c} \text{Diagram 5} \end{array} \sim \begin{array}{c} \text{Diagram 6} \end{array} \quad (5.4c)$$

These, together with (5.4a), establish the complete invariance under the Reidemeister move II .

The inverse images of (3.3) and (3.4) under $\mathcal{V}$, respectively, induce the equivalence relations

$$\begin{array}{c} \text{Diagram 7} \end{array} \sim \begin{array}{c} \text{Diagram 8} \end{array} \sim \Lambda^{-1/2} \begin{array}{c} \text{Diagram 9} \end{array} \quad (5.6a)$$

Recall that (5.6) only gives regular isotopy. In order to establish ambient isotopy the factors of $\Lambda^{\pm 1/2}$ on the right-hand side of (5.6) must be compensated. This is achieved by defining the action of $\mathcal{V}$ on the writhe number w of the tangle as

$$\mathcal{V}[w] \rightarrow (\text{multiplicative factor of}) \Lambda^{w/2} \in C[[q]]. \quad (6.1c)$$

To summarize, a $T^{(n)}$ that is a disjoint union of n open strands $\{o_i\}$ and any number of closed strands $\{c_j\}$ and has a writhe number w is mapped by $\mathcal{V}$ to $\mathcal{A}^{\otimes n}$ by

$$\mathcal{V}[T^{(n)}] = \Lambda^{w/2} \left(\prod_j \text{Tr}(\mathcal{V}(c_j)) \right) \left(\bigotimes_i (\mathcal{V}(o_i)) \right) \in \mathcal{A}^{\otimes n}. \quad (6.3)$$

By definition $\mathcal{V}$ is a homomorphism:

$$\mathcal{V}[T^{(n)} T^{(n)'}] = \mathcal{V}[T^{(n)}] \mathcal{V}[T^{(n)'}], \quad (6.4)$$

where the product on $\mathcal{V}$ is just the product on $\mathcal{A}$. We now have our first result:

Theorem 6.5. *The map $\mathcal{V}$ gives an ambient isotopy for tangles.*

The proof follows from the local properties, (5.1a-f), and global properties, (6.1a-c), of $\mathcal{V}$, and from Reidemeister's theorem [5]. $\square$

An l -component link, denoted generically by L , is a $(0,0)$ -tangle with l closed strands, therefore

Corollary 6.6. *The map $\mathcal{V}[L]$ is a link invariant.*

$\mathcal{V}$ has special properties when $n = 1$. For simplicity denote $T^{(1)}$ by T . From the invariance of Reidemeister moves *II* and *III* it follows that for any two T and T' , $\mathcal{V}[TT'] = \mathcal{V}[T'T]$ so that, from (6.4)

$$[\mathcal{V}[T], \mathcal{V}[T']] = 0. \quad (6.7)$$

Therefore,

Lemma 6.8. *The set $\{\mathcal{V}[T] \mid \text{all } (1,1)\text{-tangles } T\}$ is a commutative subset of $\mathcal{A}$.*

Then we have, from Schur's lemma,

Theorem 6.9. *Every irreducible representation of $\mathcal{V}[T]$ is one dimensional.*

In section 7 we prove a more powerful theorem:

Theorem 6.10. *$\mathcal{V}[T]$ is a central element in $\mathcal{A}$, $\forall T \in \{T^{(1)}\}$.*

Consider a splice. Call the result of replacing all its uncrossings by unsigned dots a *shadow splice*. Consider an equivalence relation whereby two tangles are equivalent if they only differ by at most the signs of the crossings, and call the equivalence classes of tangles under this relation *shadows*. It is clear that splicing a shadow obtains a shadow splice. The first of the following is obvious.

Lemma 7.1. *Splicing is a surjection from tangles to shadow splices and a bijection between shadows and shadow splices.*

Lemma 7.2. *Under splicing, the reverse image of every shadow splice contains a tangle equivalent to an untangle.*

Proof of Lemma 7.2. Consider a shadow splice S whose reverse image the shadow P is composed of a set of closed and open strands. Consider a three dimensional manifold $M^3 \sim R^2 \times Z$ in whose R^2 factor reside S and P . Let $\tilde{T}$ be a tangle in M^3 that is the disjoint union of the components of P , each of which resides in R^2 but with a distinct Z coordinate. Since there are no crossings in $\tilde{T}$, by Reidemeister's theorem its projection T on R^2 is equivalent to an untangle. By definition the shadow of T is equivalent to P . $\square$

$\mathcal{V}[T]$ is central if it commutes with all the generators $\{k_i^{\pm 1}, X_i^{\pm}\}$ of $\mathcal{A}$. We omit the proof of $[\mathcal{V}[T], k_i^{\pm 1}] = 0$, which follows trivially from the relations $[\Delta(k_i^{\pm 1}), \mathcal{R}] = 0$ and $[k_i^{\pm 1}, h^{\pm 1}] = 0$ and is a simpler version of the proof of $[\mathcal{V}[T], X_i^{\pm}] = 0$, given below. Under coproduct

$$\Delta(X_i) = X_i \otimes k_i + k_i^{-1} \otimes X_i, \quad i = 1 \dots r, \quad (7.3)$$

where r is the rank of the quantized Lie algebra. Since the index i plays no role in the proof it will be suppressed in the rest of the discussion. On a diagram, represent X by a solid circle, k (k^{-1}) by a hollow arrow pointing along (against) the orientation of the strand. Thus (2.1), with $a = X$, is diagrammatically



$$(7.4a)$$

and its companion relation $\mathcal{R}^{-1}T \circ \Delta(X) = \Delta(X)\mathcal{R}^{-1}$ is

$$\text{crossing with dot on top-left} + \text{crossing with dot on top-right} \sim \text{crossing with dot on bottom-left} + \text{crossing with dot on bottom-right} \quad (7.4b)$$

The splices of the above two relations differ only by the signs of the uncrossings, therefore they combine to give a relation for shadow splices

$$\text{shadow splice with dot on left} + \text{shadow splice with dot on right} \sim \text{shadow splice with dot on left} + \text{shadow splice with dot on right} \quad (7.5)$$

Similarly, from the relations $\Delta(k)\mathcal{R} = \mathcal{R}\Delta(k)$ one obtains

$$\text{shadow splice with dot on left} \sim \text{shadow splice with dot on right} \quad (7.6)$$

or with all the directions of the hollow arrows reversed when k is replaced by k^{-1} . These relations mean that, on a shadow splice, a pair (k, k) (or (k^{-1}, k^{-1})) may freely cross a dot, while the two pairs (X, k) and (k^{-1}, X) must move in tandem to cross a dot. Recall that any dot in these shadow splices may represent a positive or a negative crossing. Let P be the splice of the tangle T , and denote the spliced $\mathcal{V}[T]X$ and $X\mathcal{V}[T]$, respectively, by PX and XP . Similarly define Pk and kP .

Lemma 7.7. $[\mathcal{V}[T], X] = 0$ implies $PX = XP$, and vice versa.

This follows from Lemma 4.1. The lemma does not apply to higher tangles.

To see how X may be commuted through P , consider

$$\text{shadow splice with dot on left} \sim \text{shadow splice with dot on left} - \text{shadow splice with dot on right} + \text{shadow splice with dot on left} \quad (7.8)$$

obtained by combining (7.5 and 7.6). T being a $(1, 1)$ -tangle, there is only one open strand in P . By identifying the left-hand strands in the diagram on the left-

s_i . Then $\sum_i w_i/2 = w$. Define

$$\mathcal{V}_{\{\pi_i\}}[T^{(n)}] \equiv \left(\prod_i \pi_i \right) (\mathcal{V}[T^{(n)}]) = \left(\prod_{i=1}^l \lambda_{\pi_i}^{w_i/4} \right) \left(\prod_{j=n+1}^l \text{Tr}_{\pi_j}(\mathcal{V}[c_j]) \right) \left(\bigotimes_{i=1}^n \mathcal{V}_{\pi_i}[o_i] \right). \quad (8.1)$$

Theorem 8.2. $\mathcal{V}_{\{\pi_i\}}[T^{(n)}]$ is an (n, n) -tangle invariant.

This follows from (6.3) and Corollary 6.11. If $T^{(n-1)}$ is obtained from $T^{(n)}$ by closing, say, the open strand o_v , then the invariant $\mathcal{V}_{\{\pi_i\}}[T^{(n-1)}]$ is given by the right-hand side of (8.1), with the exception that the factor $(\mathcal{V}_{\pi_v}[o_v])$ is replaced by $\text{Tr}_{\pi_v}(\mathcal{V}[o_v])\text{Tr}_{\pi_v}(h) = \text{Tr}_{\pi_v}(\mathcal{V}[o_v])\text{Tr}_{\pi_v}(h^{-1})$.

9. Cabled tangle invariants

We use the coproduct in the quantum group on $\mathcal{V}$ to construct invariants for *cabled tangles*. In a cabled tangle, at least one of the strands (closed or open) is replaced by a set of parallel strands. The point is to use $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ for $\mathcal{V}[\text{strand}] \rightarrow \mathcal{V}[\text{cable}]$. Define a homomorphism $\Delta^{(u)} : \mathcal{A} \rightarrow \mathcal{A}^{\otimes u}$, where u is a positive integer, by

$$\Delta^{(u)} \equiv (1 \otimes \cdots \otimes 1 \otimes \Delta) \Delta^{(u-1)}, \quad u > 1, \quad (9.1)$$

with $\Delta^{(1)} = 1$. If $\pi \in \text{End}(V_{\mathcal{A}})$ is a fundamental representation of $\mathcal{A}$, then from the fusion rule

$$\pi^{\otimes u} = \bigoplus_k \phi_k^{(u)} \in \text{End}(V_{\mathcal{A}}^{\otimes u}) \quad (9.2)$$

irreducible representations $\phi_k^{(u)}$ of $\mathcal{A}^{\otimes u}$ are obtained. The idea of using the coproduct and the fusion rule for cabling is expressed in the following diagram

$$\begin{array}{ccc} \text{tangle} & \xrightarrow{\text{cabling}} & \text{cabled tangle} \\ \downarrow \mathcal{V} & & \downarrow \mathcal{V} \\ \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A}^{\otimes u} \\ \downarrow \pi_i & & \downarrow \phi_k^{(u)} \\ \text{End}(V_{\mathcal{A}}) & \xrightarrow{\text{fusion rule}} & \text{End}(V_{\mathcal{A}}^{\otimes u}) \end{array}$$

Define

$$\mathcal{R}_{ij}^{(u,v)} \equiv (\phi_i^{(u)} \otimes \phi_j^{(v)})(\Delta^{(u)} \otimes \Delta^{(v)})\mathcal{R}, \quad (9.3)$$

$$h_i^{(u)} \equiv \phi_i^{(u)}(\Delta^{(u)}h), \quad (9.4)$$

where in (9.3), $\mathcal{R} = e_\sigma \otimes e^\sigma$, and $\Delta^{(u)} \quad (\Delta^{(v)})$ acts on $e_\sigma \quad (e^\sigma)$. For simplicity we write

$$a' = \Delta^{(u)}(a) \in \mathcal{A}^{\otimes u}, \quad (9.5)$$

for generic $a \in \mathcal{A}$ and u . Similarly the various quantities in (9.3) and (9.4) will respectively be written as $\mathcal{R}'_{ij}$, ϕ'_i and h'_i , etc., with the understanding that, in (9.3), u and v need not be the same. The discussions in section 5 can be repeated for cables, provided the appropriate $\mathcal{R}'_{ij}$, h'_i and λ'_i are substituted for the $\mathcal{R}$, h and λ used there. We omit the discussion, except to point out that

$$\lambda'_i = \phi_i^{(u)}(\lambda) = (\lambda_i)^u, \quad (9.6)$$

which is easily understood from the diagram below.


(9.7)

This establishes the ambient invariance of the Reidemeister moves for cables. There is a significant difference between the description here and in section 5: here cables are mapped to *irreducible representations* of $\mathcal{A}^{\otimes u}$, whereas in section 5 single strands were directly mapped to $\mathcal{A}$. Let T' be a cabled tangle. By changing every cable in T' into a single strand one obtains the single-strand tangle T , for which (8.1) gives an invariant. By replacing the quantities T , Tr , λ , o_i and c_i in (8.1) by their respective cabled counterparts (denoted by primed symbols) and the set $\{\pi_i\}$ by $\{\phi_k\}$ one obtains

$$\mathcal{V}'_{\{\phi_k\}}[T^{(n)'}] \equiv \bigotimes_i \phi_i(\mathcal{V}[T^{(n)'}]) = \left(\prod_i (\lambda'_{\phi_i})^{\frac{w_i}{4}}\right) \left(\prod_j Tr'_{\phi_j}(\mathcal{V}[c'_j])\right) \left(\bigotimes_i \phi_i(\mathcal{V}[o'_i])\right). \quad (9.8)$$

where the indices i and j have the same range as in (8.1). The discussion in section 6 can be repeated to give:

Theorem 9.9. $\mathcal{V}'_{\{\phi_k\}}[T^{(n)'}]$ is an ambient isotopy for cabled tangles.

Similarly, Theorem 6.10 holds with $\mathcal{A}$ replaced by $\Delta^{(u)}(\mathcal{A}) \subset \mathcal{A}^{\otimes u}$. Therefore,

Corollary 9.10. *For any cabled $(1, 1)$ -tangle T' , $\mathcal{V}'_{\{\phi_k\}}[T']$ is proportional to the unit matrix.*

Suppose $\phi_1 \in \{\phi_k\}$ acts on the open cable in T' whose closure is L' , then the analog of (6.6) is

$$\mathcal{V}'_{\{\phi_k\}}[T'] = Q'_{\{\phi_k\}}[T']\phi_1(1), \quad (9.11)$$

so that

Corollary 9.12. $Q'_{\{\phi_k\}}[T^{(1)'}]$ is a link polynomial for the closure L' of $T^{(1)'}$.

10. Examples

We use the simplest nontrivial representations of two quantum groups, $sl(2; C)_s$ and $(gl(2; C)/U(1))_{q,s}$, to demonstrate some of the ideas discussed in the preceding sections. q and s are deformation parameters; s instead of the usual q is used for $sl(2; C)_s$ to make the comparison between the two quantum groups more transparent. Correspondingly, the generic algebraic field $C[[q]]$ referred to in previous section should be replaced respectively by $C[[s]]$ and $C[[q, s]]$ for the two quantum groups.

$sl(2; C)_s$. The generators H , X^+ and X^- are constrained by the usual defining relations [1, 13, 2]. Define $k = s^{H/2}$. The minimum set (section 2) $\mathcal{S}^\pm$ closed under coproduct are

$$\mathcal{S}^\pm = \{k^{-1}, k, X^\pm\}, \quad (10.1)$$

whose finite bases are [12]

$$\{e_\sigma\} = \{e_1 = k^{-1}, e_2 = k, e_3 = X^-\}, \quad (10.2)$$

$$\begin{aligned} \{e^\sigma\} &= \{e^1 = (s^{-1/2}k^{-1} - s^{1/2}k)/(s^{-1} - s), \\ e^2 &= -(s^{1/2}k^{-1} - s^{-1/2}k)/(s^{-1} - s), \quad e^3 = X^+\}, \end{aligned} \quad (10.3)$$

that are dual under the 2×2 representation π' :

$$\pi'(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi'(X^-) = \eta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi'(X^+) = \eta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (10.4)$$

where $\eta^2 \equiv s^{1/2}(s^{-1} - s)$. Then from (2.8) and (3.1)

$$\pi'(\mathcal{R}) = s^{1/2} \left(\begin{pmatrix} s^{-1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & s^{-1} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$\Delta(Z) = Z \otimes k_2 + k_1^{-1} \otimes Z$. The minimum sets that close under coproduct are

$$S^\pm = \{k_1^{-1}, k_2, Z^\pm\}, \quad (10.12)$$

with bases [4]

$$\{e_\sigma\} = \{e_1 = k_1^{-1}, e_2 = k_2, e_3 = Z^-\}, \quad (10.13)$$

$$\begin{aligned} \{e^\sigma\} &= \{e^1 = (s^{1/2}k_1^{-1} + s^{-1/2}k_2)/2, \\ e^2 &= s^{-1}(s^{1/2}k_1^{-1} - s^{-1/2}k_2)/2, \quad e^3 = Z^+\}, \end{aligned} \quad (10.14)$$

that are dual under the 2×2 representation π

$$\begin{aligned} \pi(H) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \pi(Z^-) &= \eta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi(Z^+) = \eta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (10.15)$$

Then

$$\begin{aligned} \pi(\mathcal{R}) &= s^{1/2} \left(\begin{pmatrix} s^{-1} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -s \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right. \\ &\quad \left. + (s^{-1} - s) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), \end{aligned} \quad (10.16)$$

$$\pi(\lambda) = s^{-1}, \quad \pi(h) = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}. \quad (10.17)$$

Note (10.16) differs from (10.5) only by one matrix element in the second term. By definition $Q_\pi[\text{unknot}] = 1$. Furthermore, it can be shown [4,18] that $Q_\pi[L]$ is the Alexander-Conway polynomial [19,20]. A peculiar property of this representation is that $\text{Tr}_\pi(h) = 0$ identically, so that $\mathcal{V}_\pi[L] = 0$ identically. This is the reason why the polynomial vanishes for any split link and the source of a well known difficulty one encounters when attempting to construct the polynomial using the Markov trace method, and explains why the Markov trace method breaks down even for the Jones polynomial at $s^2 = -1$, since at this value of s the Jones and Alexander-Conway polynomials are identical.

It is easy to verify by direct computation that the counterparts of (10.7 and 8) hold [18], with Jones replaced by Alexander-Conway. The fusion rule of π is very different from that of its counterpart π' , however [21]. Here, for generic s ,

$$\pi^{\otimes u} = \bigoplus_{k=1}^{2^{u-1}} \{2 - \dim. \text{ reps. } \phi_k\}. \quad (10.18)$$

ϕ_k are parafermionic representations [21]. $(gl(2; C)/U(1))_{q,s}$ do have $m \times m$ representations, but only at $q^2 = e^{-2i\pi/m}$. The associated polynomials form a hierarchy of Alexander-Conway polynomials [4]. Since these representations do not appear on the right-hand side of (10.18) for any u , unlike their Jones counterparts, the polynomials are not cabled versions of the fundamental Alexander-Conway polynomial.

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Note added in proof. Invariants of tangles were recently discussed in terms of category theory by Turaev [23] and Reshetikhin [24].

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