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ON SEIFERT CIRCLES AND FUNCTORS FOR TANGLES

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ABSTRACT

The properties of the Seifert circles in an oriented tangle diagram are exploited to prove a theorem that asserts that every (n, n)-tangle diagram is isotopic to a partially closed braid, and a second one that facilitates the assignment of *wrong-way* edges, one on each Seifert circle, in a tangle diagram. These result are used to identify the structure of an abstract algebra on which a functor for the isotopy of general tangles may be constructed. Any finite dimensional irreducible representation of a quasitriangular Hopf algebra is a realization of this algebra.

§0. Introduction

Since the discovery of the Jones polynomial [1], much has been said about the invariants of links, considerably less so of the invariants of tangles. Recently, Turaev [2], Reshetikhin [3] and Reshetikhin and Turaev [4] constructed invariants of tangles from solutions of the Yang-Baxter equations [5,6] and more generally from finite dimensional representations of a quasitriangular Hopf algebra. The construction in [2,3,4] was based on the relations between the category of oriented tangles [7,8] and the category of finite dimensional representations of an algebra.

In this paper we take a different approach to construct a functor for tangles. An oriented tangle is a disjoint union of n open oriented strands and a number of closed oriented strands embedded in a compact 3-manifold, with the $2n$ ends of the open strands held fixed on the boundary. We focus on what distinguishes a

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tangle from it being simply a braid, namely the existence of Seifert circles in the former and their absence in the latter. This leads us to characterize the isotopy of tangles in the classical way - in terms of the equivalence relations given by the three Reidemeister moves [9] on tangle diagrams. For oriented tangles these moves are necessarily generalized to recognize the difference, say, between a Reidemeister *II* move involving two parallel edges and one involving two antiparallel edges. Tangles and links for which the two moves are distinct are variably referred to as framed [10] or as ribbon tangles [3,4]. Here we use the generic term oriented tangles.

In our approach, the key to constructing a functor for oriented tangles is the assignment of *wrong-way* edges on a tangle diagram; wrong in the sense that all edges in a braid are right-way. Otherwise the direction of an edge is of no importance. A consequence is that the tangles are generated by only five generators: a right-way edge (↓), a wrong-way edge (⊳) on a clockwise Seifert circle, another one (⊲) on a counterclockwise circle, an overcrossing (╳) and an undercrossing (╳), as opposed to the ten generators employed in [2,3,4].

Here is a summary of the paper. A system of line, circles and arcs is defined and its properties are used to prove Theorem 1 (§1.0.), which asserts that every (n,n)-tangle (diagram) is isotopic to a partially closed braid. Alexander's theorem [11] is a corollary: every link diagram is isotopic to a closed braid. Another corollary states that the minimum number of Seifert circles in an (n,n)-tangle is the braid index of the tangle minus n. This generalizes a recent result by Yamada [12]. The proof of Theorem 1 reveals a hidden structure in tangle: the existence of two trees respectively path-connecting all the clockwise and counterclockwise Seifert circles in the tangle. This structure is exploited to prove Theorem 2 (§2.0.), which states that only Reidemeister *II* moves involving antiparallel edges are needed to isotopically transform a tangle diagram to a partially closed braid. An algorithm for (nonuniquely) assigning wrong-way edges on a tangle, one on each Seifert circle, immediately follows. Theorem 2 is used to define a tensor algebra g of an algebra $\mathcal{A}$ on which a functor for general oriented tangles is constructed, in such a way that each open strand of a tangle is mapped to one factor of $\mathcal{A}$ (Theorem 3, §3.0.). The functor is a function of the wrong-way edges, but its value is shown to be invariant under changes of assignments of wrong-way edges on the tangle. The construction of the functor appears to be less complex than the category theory employed in [2,3,4]. However, for the realization of g we have the same result: any finite dimensional irreducible representation of a quasitriangular Hopf algebra is a realization of g.

§1. Tangles and Partially Closed Braids

Alexander proved that every oriented link diagram is isotopic to a closed braid [11]. A modern rendition of the proof is given by Birman [13]. This section gives a generalization of the theorem (Theorem 1.). A theorem on the minimum number of Seifert circles in a tangle is then proven as a corollary (Corollary 1.2.).

§1.0. Every Tangle is Isotopic to a Closed Braid

Theorem 1. *Every oriented (n,n)-tangle diagram is isotopic to a partially closed braid with n strands unclosed.*

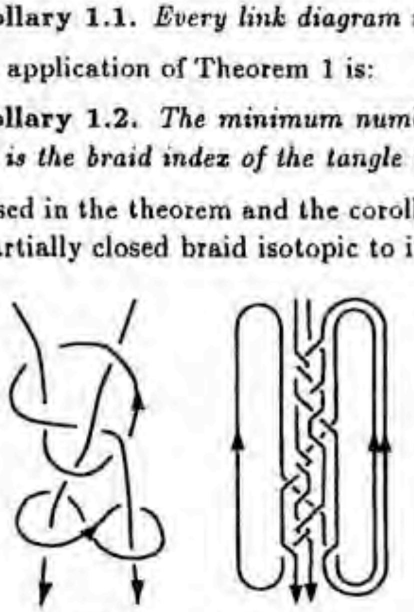
Alexander's theorem is a special case of Theorem 1:

Corollary 1.1. *Every link diagram is isotopic to a closed braid.*

Another application of Theorem 1 is:

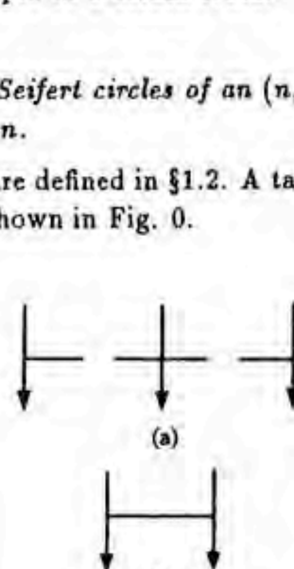
Corollary 1.2. *The minimum number of Seifert circles of an (n,n)-tangle class is the braid index of the tangle minus n.*

Terms used in the theorem and the corollaries are defined in §1.2. A tangle diagram and a partially closed braid isotopic to it are shown in Fig. 0.



(a) (b)

Figure 0



(a)

(b)

Figure 1

§1.1. A System of Lines, Circles and Arcs

Definition 1.1.1. $\{l_1, \dots l_n; C_1, \dots C_k; A_1, \dots A_m; a_1 \dots a_r\}$ in R^2 is a *system of lines, circles and arcs* where l_i is an oriented open line, C_i is a clockwise closed line called a *C-circle* and A_i is an counterclockwise closed line called an *A-circle*. The lines and circles do not intersect but are *attached* to each other via the arcs a_j. A *section* is part of an open line or a circle. An *edge* is a section delineated by two adjacent arcs. Every circle must be thus attached at least to a line or to another circle. The three configurations in which an arc may be attached to a line are shown in Fig. 1a. Two lines attached by the same arc must be parallel in the sense shown in Fig. 1b.

Remark. Fig. 2 shows an example of such a system. A similar, but not identical, system was used by Yamada [12] to prove the special case of Corollary 1.2 for $n = 0$.

Definition 1.1.2. The system is equipped with *operators* $\phi_{1,2,3}$ whose actions

$$\phi_1(s, s'; a, a') = (s, s'; a''); \qquad \phi_{1'}(s, s', s''; a, a') = (s, s's''; a''), \tag{1a}$$

$$\phi_2(s; C; \{a\}) = (s; C; \{a'\}); \quad \phi_2(s; A; \{a\}) = (s; A; \{a'\}), \tag{1b}$$

$$\phi_3(C, C'; a) = (C'; a'); \qquad \phi_3(A, A'; a) = (A'; a'), \tag{1c}$$

Lemma 1.1.2. *If the system does not have an open line, then it can be reduced to the system $\{C; A; a\}$ in Fig. 6c.*

Remark. Lemma 1.1 shall refer collectively to the three lemmas above. The reductions (and their mirror images) shown in Fig. 7, 8 and 9, respectively,

$$(s; C', C \dots; a_1, a_2 \dots) \to (s; C'; a'), \tag{2}$$

$$(C', C; a_1, a_2, a_3, a_4) \to (C'; a_1, a_2, a_3', a_4'), \tag{3}$$

$$(s; C', C; a_1, a_2, a_3, a_4) \to (s; C'; a', a_2, a_4'), \tag{4}$$

all of which involve folding, can be composed from the basic ϕ_i's. In (3) and (4), a_2 and a_4 (and correspondingly a_4'), may represent a number of arcs.

Definition 1.1.3. The arc a'' (a pair of arcs a, a', reps.) on the right- (left-) hand side of the first relation in (1a) is a *single (double) arc*. A circle is *simple* if it encircles nothing, *semisimple* if it encircles only simple circles and arcs. A *network* is the subsystem of simple A- and C-circles. A *node* is a simple circle and the arcs connected to it. A *spur* is a node connected to only one arc. A *simple node* is a node not attached to any spurs. Let S be a semisimple A-circle. A *simple network* in S is a network all of whose arcs are simple and all of whose nodes that are not spurs are simple. A *disc* is the union of a simple circle and its interior. If two sections s, s' of two different lines are attached by the arc a, then $s \cap a \neq \emptyset$ and $s' \cap a \neq \emptyset$. $\mathcal{D}$ is the region whose boundary is S; $\mathcal{D}_C$ is the union of all the C-discs in $\mathcal{D}$; $\mathcal{D}_A$ is the union of all the A-discs in $\mathcal{D}$; $\mathcal{D}_a$ is the union of all the arcs in $\mathcal{D}$; $\mathcal{D}_{Ca} = \mathcal{D}_C \cup \mathcal{D}_a$; $\mathcal{D}_{Aa} = \mathcal{D}_A \cup \mathcal{D}_a$; $\mathcal{D}_{AC} = \mathcal{D}_A \cup \mathcal{D}_C$; $\mathcal{M} = \mathcal{D}_C \cup \mathcal{D}_{Aa}$; $V = S \cup \mathcal{D}_a$. A *chain* is a sequence $(v, a, d, a', d' \cdots a'', v')$, $v \neq v' \in V$, $a \neq a' \neq \cdots \in \mathcal{D}_a$, $d \neq d' \neq \cdots \in \mathcal{D}_{AC}$. A *minimal chain* is (v, a, v'), $a \in \mathcal{D}_A$. U is the union of all minimal chains in $\mathcal{D}$. A *polygon* is a closed sequence $(\nu, \alpha, \nu', \alpha', \cdots \nu)$, $\nu \neq \nu' \neq \cdots \in V \cup \mathcal{D}_{AC}$, $\alpha \neq \alpha' \neq \cdots \in \mathcal{D}_a \cup (S \backslash V)$.

Remark. Lemmas 1.2-9 in the following refer S to a semisimple A-circle. For a semisimple C-circle, interchange C- and A- everywhere.

The rule given in Fig. 1 implies that inside S: an A-circle may be attached to either S or C-circles; a C-circle may only be attached to the simple A-circles. Hence a chain may only have alternating A- and C-discs; all polygons have at least one A-disc; and the smallest polygon not intersecting S is a rectangle with two pairs of A- and C-discs as vertices.

Lemma 1.2. *A system composed of a network in S can be reduced to a system composed of a simple network in S plus at most C-spurs attached externally to S.*

Proof. Arcs that are not single can be reduce by ϕ_1 to a single arc. Suppose there is a C-spur in $\mathcal{D}$. It must be attached to an A-node. If the smallest polygon enclosing it intersects S, then the spur can be expelled from $\mathcal{D}$ through S by ϕ_2

to become a spur attached externally to S. Otherwise the spur can be folded into a C-circle living on the polygon enclosing the spur. Suppose there is an A-spur attached to a C-node in $\mathcal{D}$. Then, since the C-node must live on a chain, the spur can be folded into an A-circle to which the C-node is attached. This leaves at most A-spurs attached by single arcs to S. □

Remark. An example of a simple network in S is given in Fig. 10. In the following we consider a simple network containing at least one C-circle enclosed by S.

Lemma 1.3. *$\mathcal{D}\backslash U = \{\mathcal{D}_i\}$ is a disconnected set of regions $\mathcal{D}_i$, and $\partial\mathcal{D}_i \subset S \cup U$, $\forall\mathcal{D}_i$.*

Proof. The minimal chains in U are the only noncontractible objects in $\mathcal{D}_{Aa}$. □

Lemma 1.4. *$\mathcal{D}_i\backslash(\mathcal{D}_i)_{Aa}$ is path connected, $\forall\mathcal{D}_i$.*

Proof. $(\mathcal{D}_i)_{Aa}$ does not contain any minimal chains. □

Lemma 1.5. *$\partial\mathcal{D}_i \cap S$ includes at least two disconnected sections of S, $\forall\mathcal{D}_i \in \{\mathcal{D}_i|(\mathcal{D}_i)_C \neq \emptyset\}$.*

Proof. Every C-disc lives on a chain that has two end points $v, v' \in V \subset S$. □

Remark. Lemmas 1.4 and 1.5 ensures that every C-disc in $\mathcal{D}\backslash\mathcal{D}_{Aa}$ is path connected to S.

Definition 1.1.4. $\forall\mathcal{D}_i$, $\{\mathcal{D}'\}_i \equiv \mathcal{D}_i\backslash\mathcal{M}_i$ is a set of disconnected subregions. A *tree* $\mathcal{P}$ is a simply connected set of 0-simplicies, or *vertices* and 1-simplicies composed of two vertices connected by a *path*. An example is shown in Fig. 11, where the circles are vertices and the dashed lines are paths.

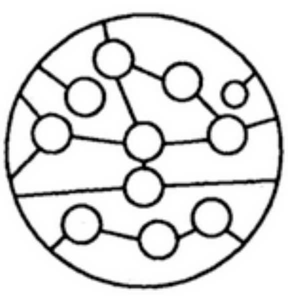


Figure 10

Figure 11

Lemma 1.6. *$\exists\mathcal{P}_{\mathcal{D}_i} \subset \mathcal{D}_i\backslash(\mathcal{D}_i)_{Aa}$, $\forall\mathcal{D}_i \in \{\mathcal{D}_i|(\mathcal{D}_i)_C \neq \emptyset\}$, such that:*

(i) $\mathcal{P}_{\mathcal{D}_i}$ has one vertex, called the root, on S that is connected by one and only one path to the rest of the $\mathcal{P}_{\mathcal{D}_i}$;

(ii) the union of all the other vertices on $\mathcal{P}_{\mathcal{D}_i}$ is $(\mathcal{D}_i)_C$.

Proof. The set $\{\mathcal{D}'\}_i$ is disconnected, but $(\{\mathcal{D}'\}_i) \cup (\mathcal{D}_i)_C = \mathcal{D}_i\backslash(\mathcal{D}_i)_{Aa}$ is path connected (Lemma 1.4). $\forall\mathcal{D}' \in \{\mathcal{D}'\}_i$ is simply connected and $\partial\mathcal{D}'$ is a polygon ($\in \mathcal{M}_i \cup S$). Select $\mathcal{D}'_1 \in \{\mathcal{D}'\}_i$ such that $\partial\mathcal{D}'_1 \cap S \neq \emptyset$ (Lemma 1.5). Select any $C_0 \in \{C_i\}_1 \equiv \partial\mathcal{D}'_1 \cap (\mathcal{D}_i)_C$. Draw a path in $\mathcal{D}'_1$ from each of the other C-discs in $\{C_i\}_1$ to C_0 and draw a path from C_0 to $\mu_i \in \partial\mathcal{D}'_1 \cap S$. Denote this set of paths by $\mathcal{P}(\mu_i; \{C_i\}_1)$. Next consider a subregion $\mathcal{D}'_2$ adjacent to $\mathcal{D}'_1$ and connected to it through one or more of the C-discs in $\{C_i\}_1$. Select any one of these C-discs and call it C_1. Let $\{C_i\}_2$ be the set of all C-discs on $\partial\mathcal{D}'_2$ not included in $\{C_i\}_1$. Draw the set of paths $\mathcal{P}(C_1;\{C_i\}_2)$ connecting all the C-discs in $\{C_i\}_2$ to C_1. In the same way obtain the set of paths for each of the other subregions $\mathcal{D}'_3\cdots$ adjacent to $\mathcal{D}'_1$. Repeat the process for each of the unprocessed subregions adjacent to $\mathcal{D}'_2$, $\mathcal{D}'_3\cdots$, and so on, until one runs out of subregions with boundaries containing C-discs not already connected by paths to the tree. Lemma 1.4 ensures that all the subregions, and therefore all the C-discs, in $\mathcal{D}_i$ will be covered by this procedure. By construction $\mathcal{P}_{\mathcal{D}_i} \equiv \mathcal{P}(\mu_i;\{C_i\}_1)\cup\mathcal{P}(C_1;\{C_i\}_2)\cup\cdots$ is a tree with root μ_i, and each C-disc in $\mathcal{D}_i$ appears as a vertex on the tree once and only once. □

Lemma 1.7. *$\exists\mathcal{D}^* \in \{\mathcal{D}'\}_i$ such that $\mathcal{D}^*\cap\mathcal{P}_{\mathcal{D}_i} = \emptyset$ and $\partial\mathcal{D}^*\cap S \neq \emptyset$, $\forall\mathcal{D}_i \in \{\mathcal{D}_i|(\mathcal{D}_i)_C \neq \emptyset\}$.*

Proof. The vertices at the tips of $\mathcal{P}_{\mathcal{D}_i}$ minus its root live on at least one chain in $\mathcal{D}_i$. The vertices are C-discs so the chain is not in $\partial\mathcal{D}_i$ (Lemma 1.3). Therefore on the side of the chain not containing $\mathcal{P}_{\mathcal{D}_i}$ are one or more subregions in $\{\mathcal{D}'\}_i$. The chain ends at two points on S, so at least one of the subregions is partly bounded by a section in S. □

Lemma 1.8. *Given the set of trees $\{\mathcal{P}_{\mathcal{D}_i}\,|(\mathcal{D}_i)_C \neq \emptyset\}$, $\exists\mathcal{P}_A \subset \mathcal{D}\backslash\mathcal{D}_{Ca}$ such that:*

(i) $\mathcal{P}_A$ has a root vertex $\mu^* \in \partial\mathcal{D}^*\cap S$ that is connected by one and only one path to the rest of the tree;

(ii) the union of the other vertices of $\mathcal{P}_A$ is $\mathcal{D}_A$;

(iii) $\mathcal{P}_A \cap (\cup\mathcal{P}_{\mathcal{D}_i}) = \emptyset$.

Proof. (i) is a consequence of Lemma 1.7. (ii) is true because $\mathcal{D}\backslash\mathcal{D}_{Ca}$ is path connected, there being no chains in $\mathcal{D}$ that does not contain at least one A-dics. (iii) is true because each $\mathcal{P}_{\mathcal{D}_i}$ contains no A-discs and is contractible to its root μ_i; $\mathcal{P}_A$ contains no C-discs and is contractible to its root μ^*; and all roots are distinct. □

Definition 1.1.5. $\mathcal{P}_{\mathcal{D}_i}$ is a *C-tree*. $\mathcal{P}_A$ is an *A-tree*. A *complete set of trees* for the simple network in S is $\mathcal{P}_A \oplus \{\mathcal{P}_{\mathcal{D}_i}\,|(\mathcal{D}_i)_C \neq \emptyset\}$. In the following we abbreviate $\{\mathcal{P}_{\mathcal{D}_i}\,|(\mathcal{D}_i)_C \neq \emptyset\}$ by $\{\mathcal{P}_{\mathcal{D}_i}\}$.

Remark. Fig. 12a shows a network with a complete set of trees. Here and in figures in the rest of the paper, where confusion may arise, arcs are drawn in heavy lines. In Fig. 12b the arcs on the network are deleted to show the trees more clearly. The complete set of trees contains two C-trees with roots μ_1 and μ_2, respectively, and the single A-tree with root μ^*.

Lemma 1.9. *A simple network in S can be reduced to the form shown in Fig. 13a. The number of new C-spurs around the outer perimeter of S in its reduced form is equal to the number of $\mathcal{D}_i$'s in $\{\mathcal{D}_i|(\mathcal{D}_i)_C \neq \emptyset\}$.*

Proof. From Lemmas 1.6-8 there exists a complete set of trees $\mathcal{P}_A \oplus \{\mathcal{P}_{\mathcal{D}_i}\}$ for the network. Let A_0 be the A-circle to which the root μ^* of the only A-tree $\mathcal{P}_A$

and one simple network each to the left of open line l_1 and to the right of the open line l_n. For the network on the right, close the open line l_n by identifying both of its ends with the point μ_n^* at infinity to the right to create, by virtue of the simple network it now encloses, a semisimple A-circle S_n^*. Use Lemma 1.9 to reduce S_n^* to the form of Fig. 13a, where now S_n^* encloses a single A-circle A_0. In the reduction, choose μ^* as the root of the A-tree $\mathcal{P}_A$. Denote by S^* the subsystem composed of S_n^* and A_0 and for the moment think of it as a simple A-circle.

The system to the right of the open line l_{n-1} is now at most a simple net work. Again close the line by identifying its ends with μ_{n-1}^* at infinity to the right to form a semisimple circle S_{n-1}^* and use Lemma 1.9 to reduce it to the form of Fig. 13a as before, except to choose μ_{n-1}^* to be the root of $\mathcal{P}_A$ and choose μ_n^* to be the only vertex on $\mathcal{P}_A$ connected to μ_{n-1}^*. This means that S_{n-1}^* has only S^* inside. S^* contains all other the A-circles previously in S_{n-1}^*. Fold all these A-circles into A_0 and call the result A_0, fold S^* into S_{n-1}^* and identity μ_n^* with μ_{n-1}^* in the process, then denote by S^* the new subsystem of S_{n-1}^* containing A_0.

Repeat this process until all the open lines are exhausted. The system now is composed of S^* enclosing a simple A_0, and possibly a simple network attached externally to S^*. All the open lines in the original system have been folded into S^*. By viewing the system inside-out relative to ∂S^*, Lemma 1.9 can be used once more to reduce the simple network so that all the C-circles are folded into one to yield C and all the A-circles are expelled *into* S^*, which can then be folded into A_0 to yield A. The system is now composed of S^* attached internally to A and externally to C. Since nothing has happened to the edge containing μ^* (Lemma 1.10), an open line l is recovered by opening S^* at μ^*. The system now has the form of Fig. 6a.□

Proofs of Lemmas 1.1.1 and 1.1.2. These are just trivial specializations of Lemma 1.1.1. □

§1.2. Tangle as a System of Lines, Circles and Arcs

In order to use Lemma 1.1 (*i.e.*, the set of lemmas 1.1.0, 1.1.1 and 1.1.2) to prove Theorem I, we need to interpret the abstract objects in the system as objects on a tangle.

Definition 1.2.0. An *oriented* (n,n)-*tangle* T is the disjoint union of n open oriented strands and a number of oriented closed strands embedded in a cylinder in a 3-manifold, with all the n tails of the open strands held fixed on the ceiling of the cylinder and all the n tips held fixed on the floor. A *link* is a $(0,0)$-tangle.

Remark. In the follow the word "oriented" will normally be suppressed. The above definition of (n,n)-tangles can be generalized to describe any tangle. Consider a tangle of n open strands (and a number of closed strands) embedded in a 3-ball with an S^2 boundary. The $2n$ ends of the tangle are held fixed on S^2. We consider manifolds on which any two points on the S^2 are path connected. Then it is possible to draw a circle on the S^2, *without moving any of the ends on* S^2, that delineates all

Definition 1.2.4. A *braid* is a tangle diagram whose splice does not have any Seifert circles. An (n,n)-*braid-tangle* is the result of closing m strands of an $(n+m)$-strand braid.

Remark. Since the action of closing a strand in a braid must not generate any new crossing, an (n,n)-braid-tangle is obtained from an $(m+n)$-strand braid by closing the m_1 left most strands clockwise and the m_2 right most strands counterclockwise, $m_1 + m_2 = m$.

Definition 1.2.5. The three *Reidemeister moves* [9] *I*, *II* and *III* in splice presentation are shown in Fig. 17a, 17b and 17c.

Remark. For the Reidemeister *III* move, label the three signs, from top to bottom, by (α,β,γ) in each of the two diagrams in Fig. 17c, and use the subscripts L and R, respectively, to denote the left and right diagrams. Then a precise description of the relation is $(\alpha,\beta,\gamma)_L = (\gamma,\beta,\alpha)_R$. Of the eight possible sets of signs, the two sets $(+,-,+)$ and $(-,+,-)$ are excluded from the relation.

Definition 1.2.6. A Reidemeister move *IIb* is a type of Reidemeister *II* move involving two antiparallel strands.

Remark. The splice presentation of such a move is shown in Fig. 18a.

Definition 1.2.7. The *writhe* and the *Seifert number* are, respectively, the number of positive signs minus the number of negative signs and the number of Seifert circles on a splice.

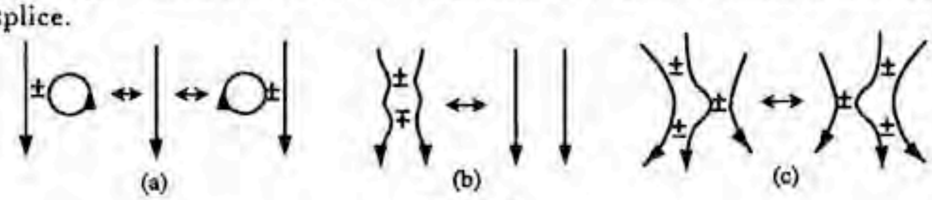


Figure 17

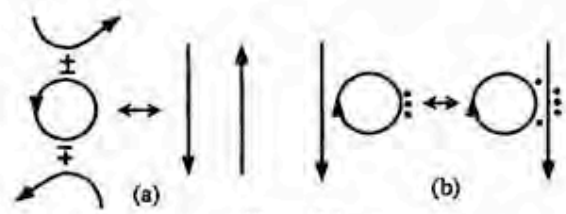


Figure 18

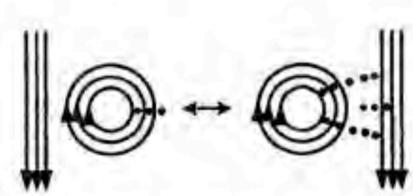


Figure 19

Remark. In the following we reserve the terminology Reidemeister *II* to mean only those moves involving two parallel lines. It is evident that in a splice neither the writhe nor the Seifert number is an invariant of isotopy. However, only Reidemeister moves that *do* preserve the writhe *and* the Seifert number are needed to prove Theorem 1 (see below). First observe that move *I* preserves neither the writhe nor the Seifert number, whereas moves *II* and *III* preserve both. Move *IIb* preserve writhe, but does not necessarily preserve Seifert number.

Definition 1.2.8. A Reidemeister move *IIb*1 involves two lines not belonging to the same Seifert circle; a move *IIb*2 involves two lines belonging to the same Seifert

Corollary 2.2. *On every tangle there exists at least one assignment of wrong-way edges, one edge for each Seifert circle in the tangle, such that:*

(i) the identities of the wrong-way edges are preserved in the transformation;

(ii) the set of wrong-way edges coincides with the set of edges that close the strands on the partially closed braid.

Figure 20

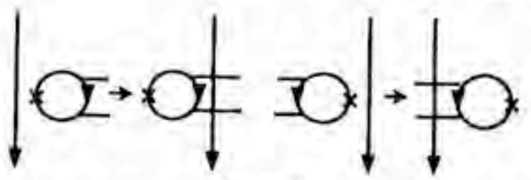


Figure 21

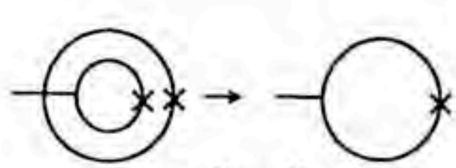


Figure 22

§2.1. A System with a Distinguished Point on Each Circle

Definition 2.1.1. Given a system $\mathcal{Y} = \{l_1, \ldots; C_1, \ldots; A_1, \ldots; a_1 \ldots\}$, a system $\hat{\mathcal{Y}} = \{l_1, \ldots; \hat{C}_1, \ldots; \hat{A}_1, \ldots; a_1 \ldots\}$ is identical to $\mathcal{Y}$ except that in $\hat{\mathcal{Y}}$ there is one and only one *distinguished point* on every circle. $\hat{C}$ and $\hat{A}$ denote circles with distinguished points. In analogy to (1), the actions of the *restricted operators* $\hat{\phi}_{1,2,3}$ on $\hat{\mathcal{Y}}$ are explained in Figs. 20-22, where each distinguished point is marked by a $\times$ symbol. Actions corresponding to those in Figs. 7-9 are given in Figs. 23-25.

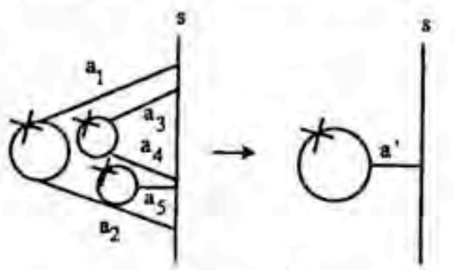


Figure 23

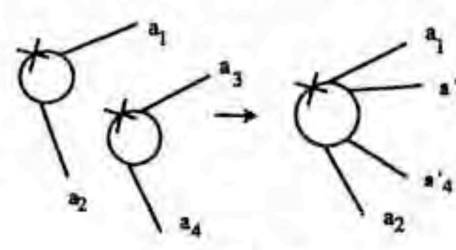


Figure 24

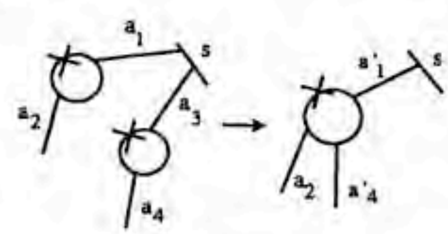


Figure 25

Lemma 2.1. *If the operators $\hat{\phi}_i$ are invertible, then $\exists\hat{\mathcal{Y}}$ such that Lemma 1.1 is true, $\forall\mathcal{Y}$.*

The cycle Ω_m reduces some semisimple circle $(S, W; \{a\})$ in $\mathcal{Y}_{m-1}$ to a node $(S, \{C_i\}, \{a\})$ in $\mathcal{Y}_m$ (replace C by A if S is a C-circle). Lemma 2.2 assures that if all multiple arcs in the semisimple circle $(\hat{S}, \{\hat{C}_i\}, \{a\})$ are reducible by $\hat{\phi}_1$ to single arcs, then there exists an inverse cycle $\hat{\Omega}_m^{-1}$ that causes the node $(\hat{S}, \{\hat{C}_i\}; \{a\})$ to grow into the semisimple circle $(\hat{S}, \{\hat{C}_i\}, \{a\})$, all of whose multiple arcs are reducible to single arcs. The system $\hat{\mathcal{Y}}_N$ is unique since there is only one way to fix the distinguished points on C and A in $\mathcal{Y}_N$. The open line l in $\hat{\mathcal{Y}}_N$ is equivalent to an A-circle with distinguished point at infinity to the right (See proof of Theorem 1; it is also equivalent to a C-circle with distinguished point at infinity to the left.). Identify this A-circle with S. Then $\mathcal{Y}_N$ has the form $(S, C, \emptyset)$, and correspondingly $\hat{\mathcal{Y}}_N$ has the form $(\hat{S}, \hat{C}, \emptyset)$, which is a node attached to one spur by a single arc. Thus, by Lemmas 2.2 and 2.3, two sequences $\hat{\Omega}_N^{-1}, \hat{\Omega}_{N-1}^{-1}, \ldots, \hat{\Omega}_1^{-1}$ and $\hat{\mathcal{Y}}_N, \hat{\mathcal{Y}}_{N-1}, \ldots, \hat{\mathcal{Y}}_1, \hat{\mathcal{Y}}$, in the orders given and satisfying the growth sequence

$$\hat{\mathcal{Y}}_N \xrightarrow{\hat{\Omega}_N^{-1}} \hat{\mathcal{Y}}_{N-1} \xrightarrow{\hat{\Omega}_{N-1}^{-1}} \cdots \hat{\mathcal{Y}}_1 \xrightarrow{\hat{\Omega}_1^{-1}} \hat{\mathcal{Y}}, \tag{2.3}$$

can be constructed from (2.2). Thus the system $\hat{\mathcal{Y}}$ exists and is reduced by the sequence $\hat{\Omega}_1, \hat{\Omega}_2, \ldots, \hat{\Omega}_N$ to $\hat{\mathcal{Y}}_N$. □

Definition 2.1.5. $\mathcal{T}_C$ ($\mathcal{T}_A$, resp.) is the union of the *distinguished points* on all the C-circles (A-circles) in $\hat{\mathcal{Y}}$ and all the paths traced by them in the growth sequence (2.3).

Remark. Since growth is a deformation, the paths traced by any point in the system are simply connected.

Lemma 2.4. *$\mathcal{T}_C$ ($\mathcal{T}_A$, resp.) is a tree whose set of vertices is the union of the distinguished points on the C-circles (A-circles) in $\hat{\mathcal{Y}}$, and $\mathcal{T}_C \cap \mathcal{T}_A = \emptyset$.*

Proof. In the following, X is either C or A. Denote by $\mathcal{T}_{X,m}$ the X-type $\mathcal{T}$ in $\hat{\mathcal{Y}}_m$ in the sequence (2.3). Then $\mathcal{T}_{X,N}$ consists of a single distinguished point on the only X-circle in $\hat{\mathcal{Y}}_N$. Call this point the root X_0 of $\mathcal{T}_X$. Denote by $Tip_{X,m}$ the set of vertices on $\mathcal{T}_{X,m} \backslash X_0$ each of which is connect by one and only one path to the rest of $\mathcal{T}_{X,m}$. The action of the inverse cycle $\hat{\Omega}_m^{-1}$ in (2.3) on $\hat{\mathcal{Y}}_m$ is to send $\mathcal{T}_{X,m}$ to $\mathcal{T}_{X,m-1}$ by causing each vertex in a subset $\mathcal{Q}_X \subset Tip_{X,m}$ to grow into a tree whose root is the vertex. This is the reverse of the process described in the proofs of Lemma 1.9 and Corollary 1.9.1. In fact, if $\hat{\Omega}_m^{-1}$ causes a node in $\hat{\mathcal{Y}}_m$ to grow into a semicircle S in $\hat{\mathcal{Y}}_{m-1}$, then

$$\bigcup_{X=C,A} \left((\mathcal{T}_{X,m-1} \backslash \mathcal{T}_{X,m}) \cup \mathcal{Q}_X \right) = \mathcal{P}_S,$$

where $\mathcal{P}_S$ is the complete set of trees in S.

If follows that if $\mathcal{T}_{X,m}$ is a tree then $\mathcal{T}_{X,m-1}$ is a tree, and that

$$\mathcal{T}_{C,m} \cap \mathcal{T}_{A,m} = \emptyset \Rightarrow \mathcal{T}_{C,m-1} \cap \mathcal{T}_{A,m-1} = \emptyset.$$

$\mathcal{T}_{C,N}$ and $\mathcal{T}_{A,N}$ are trees and $\mathcal{T}_{C,N} \cap \mathcal{T}_{A,N} = \emptyset$. Therefore, by induction, $\mathcal{T}_C$ and $\mathcal{T}_A$ are trees and $\mathcal{T}_C \cap \mathcal{T}_A = \emptyset$. By construction $\mathcal{T}_X$ includes every distinguished point on the X-circles in $\hat{\mathcal{Y}}$ once and only once. □

Remark. Lemma 2.4 is a heuristically obvious consequence of the sequence (2.3): $\mathcal{T}_X$ is a tree grown from X_0.

§2.2. Proof of Theorem 2

Let the system $\mathcal{Y}$ in (2.2) be a tangle system. Then $\mathcal{Y}$ corresponds to the splice of a tangle, the circles in $\mathcal{Y}$ are Seifert circles in the splice and the operators $\phi_{1,2,3}$, being composed of Reidemeister moves, are invertible. For simplicity we identify $\mathcal{Y}$ with the splice. The system $\hat{\mathcal{Y}}$ in (2.3) is identified with the same splice except that on every Seifert circle there is now a distinguished point. Lemmas 2.1 and 2.4 assert that there exists two trees, $\mathcal{T}_C$ and $\mathcal{T}_A$, along which, using the reverse of the sequence (2.3), all the n_C (n_A, resp.) clockwise (counterclockwise) Seifert circles in the splice can be bunched into one cabled clockwise (counterclockwise) Seifert circle of n_C (n_A) cables in the splice of the braid-tangle $\hat{\mathcal{Y}}_N$. Each of the cycles $\hat{\Omega}_m$ in (2.3) is itself a product of $\hat{\phi}_i$'s, as expressed in (2.1). The actions of $\hat{\phi}_1$ and $\hat{\phi}_3$ are book-keeping in nature; they just transform a braid into another braid in the same class. Lemmas 2.2 and 2.3 and the remark following Lemma 2.3 make clear that these actions neither impede nor depend on any of the actions of the $\hat{\phi}_2$ operators in the cycle, which are essential. This means that all the operators $\hat{\phi}_1$ and $\hat{\phi}_3$ may be eliminated from the cycle. Thus the total action in (2.3) can be reduced to a product of only $\hat{\phi}_2$'s. The action of $\hat{\phi}_2$ is just the Reidemeister moves *IIb*1 (Fig. 18), while the omitted actions of $\hat{\phi}_1$ and $\hat{\phi}_3$ are composed of Reidemeister moves *II* and *III* (Fig. 17b,c). □

Remark. This is a generalization of a result by Vogel [17] for links.

§2.3. Tangle with Wrong-Way Edges Specified

Definition 2.3.1. Let $\hat{\mathcal{Y}}$ tangle system with distinguished points. There is a one-to-one correspondence between the edges in $\hat{\mathcal{Y}}$ and the edges in the splice of the tangle with which $\hat{\mathcal{Y}}$ is identified. Every edge in the splice that corresponds to an edge in $\hat{\mathcal{Y}}$ on which there is a distinguished point is a *wrong-way edge*.

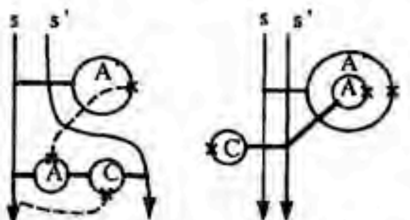
Remark. By Definition 2.1.5 and Lemmas 1.10 and 2.4, every Seifert circle in a splice has one and only one wrong-way edge. Since a braid is a tangle that does not have any Seifert circle, this implies that a braid may not have any wrong-way edge. In other words, if there is a wrong-way edge in a tangle, then it can not be isotopically transformed to a braid.

Proof of Corollary 2.2. Lemma 2.4 asserts that the distinguished points in the system are preserved in the transformation induced by (2.3) as well as in the inverse transformation. Lemma 1.10 asserts that every wrong-way is associated with a distinguished point. On a braid-tangle the wrong-way edges are just the edges that partially close the braid. □

Remark. Theorem 2 suggests the following algorithm for transforming any tangle to a braid-tangle using only Reidemeister $IIb1$ moves.

(i) Splice the tangle to get $\mathcal{Y}$.
(ii) Use the two sequences (2.2) and (2.3) to construct $\hat{\mathcal{Y}}$ from $\mathcal{Y}$ (this is equivalent to identifying all the wrong-way edges on $\mathcal{Y}$) and the two trees $\mathcal{T}_C$ and $\mathcal{T}_A$.
(iii) Pull the root of $\mathcal{T}_C$ ($\mathcal{T}_A$) to the far left (right) and the tree along with it, keeping the whole tangle rigid except to allow only those sections of a line in the immediate neighborhood of the distinguished point on each wrong-way edge to deform.
(vi) The result is a braid tangle.

A simple example is given in Fig. 28, where the tangle of Fig. 0a is transformed to the partially closed braid in Fig. 0b. In Fig. 28a a set of two trees is constructed on the system (Fig. 16c) corresponding to the splice (Fig. 16a) of the tangle. In Fig. 28b the A-circle A' passes through the open line s' and then gets folded into A', and the C-circle C passes through the open line s. Every time a Seifert circle passes through a line two more crossings are generated. This explains why the braid in Fig. 28b/Fig. 0b has 17 crossings, six more than the number of crossings in the original tangle. On the other hand, the writhe (-1) and number of Seifert circles (3) of the system, as well as the identities of the wrong-way edges marked by × symbols, are not changed.



(a) (b)
Figure 28

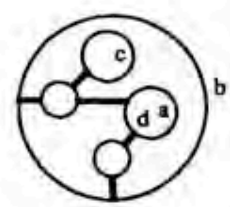


Figure 29

In the following we derive several Lemmas that will be used in the next section.

Definition 2.3.2. For a system $\mathcal{Y}$, we speak of constructing a set of trees $\mathcal{P}$ (Definition 1.1.5.) on $\mathcal{Y}$, and refer to $\hat{\mathcal{Y}}$ as $\mathcal{Y}$ equipped with $\mathcal{P}$, or as the *treed system* associated with $\mathcal{Y}$. It is understood that $\hat{\mathcal{Y}}$ includes the set of distinguished points that comes with $\mathcal{P}$, and that the latter is finite. In a drawing of a treed system $\hat{\mathcal{Y}}$, the distinguished points are always given explicitly whereas the paths

$$AhB = B'hA' = Bh'A = A'h'B' = 1; \tag{3.4}$$

$$Tr(c_1c_2\cdots c_n) = Tr(c_2\cdots c_nc_1), \qquad \forall c_1, c_2, \ldots, c_n \in \mathcal{A}; \tag{3.5}$$

where $A\cdots B$ is the shorthand for $\sum_i a_i\cdots b_i$; similarly for $B'\cdots A'$; $\mathcal{T}\circ(a\otimes b) = b\otimes a$; $\mathcal{R}_{12} = \mathcal{R}\otimes 1$; $\mathcal{R}_{13} = A\otimes 1\otimes B$; $\mathcal{R}_{23} = 1\otimes\mathcal{R}$. The multiplication m^g in g is understood as a morphism, $m^g: \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, $\mapsto m^g(a\otimes b)\equiv ab$, a, $b\in\mathcal{A}$. By (3.2) we define $\mathcal{R}^{-1} = \mathcal{T}\circ\mathcal{R}'$, $\mathcal{R}'^{-1} = \mathcal{T}\circ\mathcal{R}$.

Remark. The condition $h\neq 1$ is necessary for g to carry a functor for oriented tangles, in the sense of the *ribbon graphs* of Reshetikhin and Turaev [4].

§3.2. The Functor $\mathcal{V}$

Definition 3.2.1. For an oriented tangle T, the *specified* tangle T^* associated with T denotes T equipped with a specific choice of wrong-way edges. Call a wrong-way edge on a counterclockwise Seifert circle an *A-edge* (↾ or ⇃), and one on a clockwise circle a *C-edge* (↿ or ⇂). An edge that is neither an A-edge nor a C-edge is an *N-edge* (↑ or ↓). As a rule, the property of an edge does *not* depend on which *direction* it points to.

Definition 3.2.2. Let α, β, ... be a set of *arrows*. The tensor product $\alpha\otimes\beta$ is the disjoint union of the two arrows. The multiplication, or composition, of two arrows α and β, denoted by $\alpha\beta = m(\alpha\otimes\beta)$, is the action of joining the tip of α to the tail of β. (This is the multiplication of a groupoid.) An (open) strand τ is either a single arrow or a composition of more than one arrows by multiplication. The notions of multiplying and tensoring arrows extend to strands. m^{ij} acting on a tensor means right-multiplying the i^{th} factor of the tensor by the j^{th} factor. For example, $m_{13}(\alpha\otimes\beta\otimes\tau) = \alpha\tau\otimes\beta$, $m_{31}(\alpha\otimes\beta\otimes\tau) = \tau\alpha\otimes\beta$. The closing of a strand τ, denoted by $\langle\tau\rangle$, is the operation of joining the tip of the strand to its own tail.

Remark. From the definition above we can view a specified tangle T^* as a disjoint union of open and closed strands generated by the set of arrows ⇃, ⇂ and ↓, and pairs of tensored arrows ⤫ (overcrossing) and ⤬ (undercrossing), equipped with the operations multiplication, tensor product and closing.

Definition 3.2.3. Let h, h', $\mathcal{R}$, $\mathcal{R}'$ and Tr be the objects in g defined §3.1. Let α, β be strands in T^*. Define the map $\mathcal{V}: T^*\to g$ by

$$\mathcal{V}(\downarrow) = \mathcal{V}(\uparrow) = 1, \quad \mathcal{V}(↾) = \mathcal{V}(⇃) = h, \quad \mathcal{V}(↿) = \mathcal{V}(⇂) = h'; \tag{3.6}$$

$$\mathcal{V}(⤫) = \mathcal{R}, \qquad \mathcal{V}(⤬) = \mathcal{R}'; \tag{3.7}$$

$$\mathcal{V}(\alpha\otimes\beta) = \mathcal{V}(\alpha)\otimes\mathcal{V}(\beta); \tag{3.8}$$

$$m(\mathcal{V}(\alpha\otimes\beta)) = m^g(\mathcal{V}(\alpha)\otimes\mathcal{V}(\beta)); \tag{3.9}$$

$$\mathcal{V}(\langle\alpha\rangle) = Tr(\mathcal{V}(\alpha)). \tag{3.10}$$

which implies Fig. 34, are derived from (3.2-4).

Proof. This is proved diagrammatically in Fig. 35. □

Definition 3.3.2. Define the *transformations* in Figs. 33*a*, 33*b*, 33*c* and 34, respectively, as Reidemeister moves *II*, *III*, *I* and *IIb*, respectively, on a specified tangle T^*.

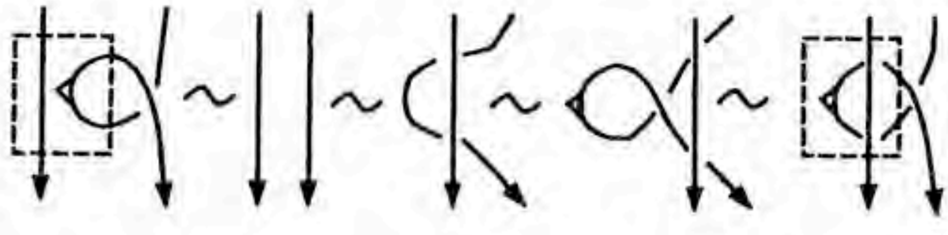


Figure 35

Remark. The Reidemeister moves are reversible. By virtue of the relations (3.2-5) and (3.11), specified tangles related by Reidemeister moves are equivalent. However, it has not yet been shown that given a Reidemeister move on T, there exists a corresponding Reidemeister move on some T^* associated with T.

For a tangle T and a specified tangle T^*, associate with T a tangle system $\mathcal{Y}$ and with T^* a tangle system with distinguished points $\hat{\mathcal{Y}}$ such that, for the latter, sections on $\hat{\mathcal{Y}}$ with a distinguished point correspond to wrong-way edges on T^*. The actions of the restricted operators $\hat{\phi}_i$ on $\hat{\mathcal{Y}}$ correspond to sequences of II, III and $IIb1$ moves on T^*. The operators are invertible because the Reidemeister moves are reversible. Denote by $\hat{\Omega}$ a product of $\hat{\phi}_i$'s or their inverses. If $\hat{\mathcal{Y}}_m$ is the system for T^*_m, $m = 1$ and 2, and $\hat{\mathcal{Y}}_1 = \hat{\Omega}\hat{\mathcal{Y}}_2$, then it follows from the preceding definition and discussion that $T^*_1 \sim T^*_2$.

Lemma 3.1. *If T^*_1 and T^*_2 are two different specified tangles of the same T, then $T^*_1 \sim T^*_2$.*

Proof. From the definition of wrong-way edges given in §2.3 and Theorem 2, only Reidemeister $IIb1$ moves are needed to transform both T^*_1 and T^*_2 to the same T^*_B, where T_B is a braid-tangle. Therefore, $T^*_1 \sim T^*_B \sim T^*_2$. □

Definition 3.3.3. Let T and T_1 be related by one Reidemeister I move such that, relative to T, T_1 has one extra crossing X and one extra Seifert circle, C. We write $T_1 = T \cup X \cup C$.

Lemma 3.2. *For any pair of specified tangles T^* and T^*_1, respectively, of T and T_1, respectively, $T^* \sim T^*_1$.*

We first prove the following.

Lemma 3.3 *For any choices of wrong-way edges T^* on T, there exists a* the two systems to which 43*a* and 43*b* respectively belong are equivalent. On the other hand, 43*b* is directly obtained from 43*a* via one of the relations in Fig. 42. Thus the two subsystems are by themselves directly equivalent.

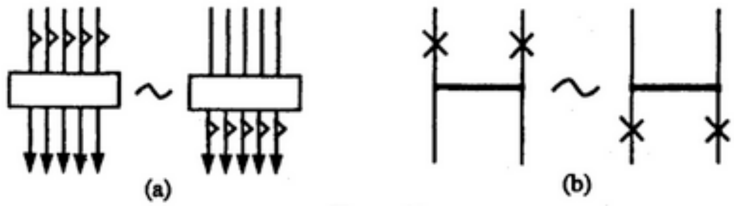


Figure 41

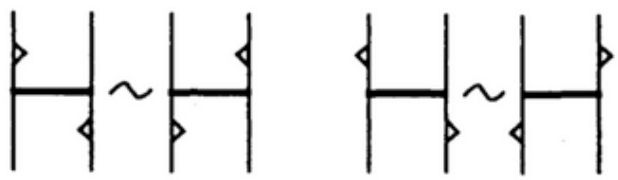


Figure 42

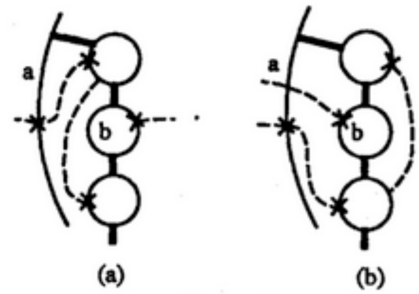


Figure 43

Let T be an (n,n)-tangle. There are $n+1$ ways of closing it: closing the m left-most strands clockwise on the left, and the rest of the strands counterclockwise on the right, $m = 0, 1, \ldots, n$. Refer to the links thus obtained by $\hat{T}_m$. Clearly all $\hat{T}_m$'s belong to the same isotopy, $[\hat{T}]$, the generic closure of T. On the functor, this relation is expressed as follows. Define

$$H_m \equiv h^{\otimes m} \otimes h'^{\otimes(n-m)} \in \mathcal{A}^{\otimes n}, \qquad m = 0, 1, \ldots, n. \tag{3.15}$$

Suppose that after closing the tangle, $\langle H_m \mathcal{V}[T] \rangle \in \mathcal{A}^{\otimes l}$. Then the following relation must hold:

$$\mathcal{V}[\hat{T}] = \mathcal{V}(\hat{T}_m) = Tr^{\otimes l}(\langle H_m \mathcal{V}[T] \rangle), \qquad \forall m. \tag{3.16}$$

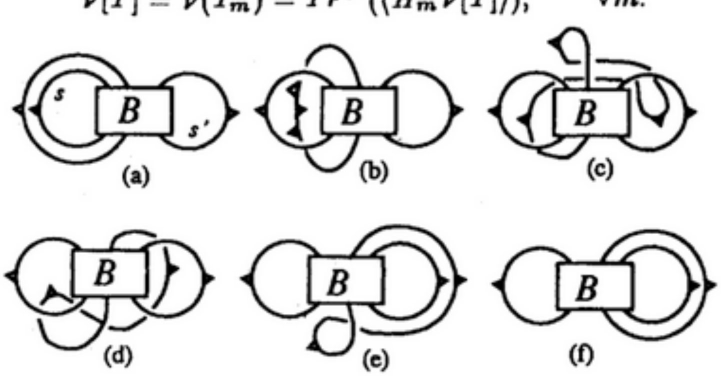


Figure 44

This relation follows from Theorem 1 and the lemma:

Lemma 3.11. *If $\mathcal{B}$ is a braid of n strands whose closure $\hat{\mathcal{B}}$ is an l-component link, then*

$$\mathcal{V}[\hat{\mathcal{B}}] = Tr^{\otimes l}(\langle H_m \mathcal{V}[\mathcal{B}] \rangle), \qquad \forall m. \tag{3.17}$$

Proof. This is given diagrammatically in Fig. 44 for the case $n = 3,\ m = 2$. The proof is extended to the general case by replacing the edges in 44a marked by s and s' by the appropriate cables. □

§3.6. Representations of A Quasitriangular Hopf Algebra

The construction of a functor for tangles on a quasitriangular Hopf algebra from the point of view of category theory was discussed in detail in [3,4]. Here we discuss briefly how the algebra g is realized as a finite dimensional irreducible representation of a quasitriangular Hopf algebra.

For notation we refer to [3,4] and Drinfel'd [14]. A quasitriangular Hopf algebra is a noncommutative and noncocommutative Hopf algebra $\{\mathcal{A},\ S,\ m,\ \triangle,\ \epsilon\}$ equipped with an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ satisfying:

$$(\triangle \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (id \otimes \triangle)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12},$$

$$(\mathcal{T} \circ \triangle(\alpha))\mathcal{R} = \mathcal{R}\triangle(\alpha), \qquad \forall \alpha \in \mathcal{A}. \tag{3.18}$$

In the Hopf algebra, $\mathcal{A}$ is an algebra over a ring $\mathcal{K}$, the antipode $S: \mathcal{A} \to \mathcal{A}$ is an antiautomorphism, and the multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, the comultiplication $\triangle: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon: \mathcal{K} \to \mathcal{A}$ are morphisms. The quantized universal enveloping algebras [14,15] of simple Lie algebras are the earliest known realizations of quasitriangular Hopf algebras. In the following, we refer to such algebras as $\mathcal{H}$.

By definition $\mathcal{R}$ satisfies (3.2) provided one identifies $\mathcal{R}' \equiv \mathcal{T} \circ \mathcal{R}^{-1}$. From the properties of $\mathcal{H}$ and (3.18), $\mathcal{R}$ satisfies (3.3). Define $v \equiv m((id \otimes S)\mathcal{R}) \in \mathcal{A}$. Then $\lambda \equiv vS(v)$ is a central element of $\mathcal{A}$ [16]. By Schur's Lemma, λ is proportional to the identity in any finite dimensional irreducible representation of $\mathcal{A}$. Let π be such a representation. Then the algebra g of §3.1 is realized by the π image of $\mathcal{H}$, provided the set of objects $\{h,\ h',\ \mathcal{R},\ \mathcal{R}',\ Tr\}$ in §3.1 are identified with the set $\{w^{-1/2}\pi(v),\ w^{-1/2}\pi(S(v)),\ \pi(\mathcal{R}),\ \pi(\mathcal{T} \circ \mathcal{R}^{-1}),\ Tr_\pi\}$, where $w = \pi(\lambda)$ and Tr_π is the matrix trace of $\mathcal{A}$ in π. In general $\pi(h) \neq 1$. This causes nontrivial complications in the construction of the functor, and is the reason why objects such as systems of distinguished point $\hat{\mathcal{Y}}$ and specified tangles T^*, in addition to systems $\mathcal{Y}$ and tangles T, were introduced. In this case $\mathcal{V}$ is a functor for framed [10], or ribbon tangles [4], and a C-edge (an A-edge) corresponds to a section of a ribbon with one full clockwise (counterclockwise) twist. See in particular [4] for a graphical description of ribbons.

This work was partly carried out at the Research Institute for Mathematical Sciences, Kyoto University. The arthur thanks the Institute for support and hospitality. He also thanks Peter Leivo for critical comments given during the early phase of this project and Toshie Takata for bringing [12] to his attention.

Note Added: After the completion of the present manuscript, a paper by Vogel [17] was brought to the attention of the author, in which, based on an idea very