Q-DEFORMATION OF $s\ell(2,\mathbb{C})\times Z_N$ AND LINK INVARIANTS*

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1. INTRODUCTION

It is by now well known[1-3] that for each simple Lie algebra or Kac-Moody algebra $\mathfrak{g}$ there is a quantum group $\mathcal{U}_q(\mathfrak{g})$, and that for each representation of this group, a knot invariant of the Jones type can be defined. Thus the Jones polynomial[4] corresponds to the 2×2 matrix representation of $\mathcal{U}_q(s\ell(2,\mathbb{C}))$ and the Akutsu-Wadati polynomial[5], the 3×3 matrix representation. Interestingly enough, the oldest knot polynomial, the Alexander-Conway polynomial[6], is yet to be identified with a quantum group. On the other hand, it is already known that the Alexander-Conway polynomial is given by a two-state solution of the quantum Yang-Baxter equation. Recently, Kauffman[7] derived the polynomial from a state model, and discovered a bialgebra associated with it. Lee[8] has shown that the polynomial at least is associated with a pseudo Hopf algebra - the representation for the universal $\mathcal{R}$-matrix was given, but the abstract form of the antipode of the algebra was not.

Here we show that the Alexander-Conway polynomial is given by the 2×2 representation of the quantum group: $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_N)$ There seems to be an infinite family of quantum groups obtained by extending $\mathcal{U}_q(s\ell(2,\mathbb{C}))$ this way: $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_N)$. We also give an explicit representation for $N=3$, which gives the Alexander-Conway type polynomial derived previously by Lee & Couture[9].

A characteristic of an Alexander-Conway type link invariant is that it has as its kernel the set of all split links[7]. This property is not shared by Jones type invariants defined on $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra. In fact, the standard Markov trace of a braid group representation constructed from the universal $\mathcal{R}$-matrix of $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_N)$

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damental group of the configuration space of n indistinguishable particles in two dimensions. Braid group representations are related to strange statistics (neither Bose or Fermi) in quantum mechanics in 2+1 dimensions [9-12] and quantum field theory in 1+1 dimensions [13] and to the monodromy of multipoint correlation functions in two dimensional conformal field theory [14,15].

In an earlier paper [12], we proposed an algorithm for constructing N-state representations of B_n of the maximally symmetric type (defined in section 2); from this family of representations one can derive an infinite sequence of link polynomials. The N=2 case in this family corresponds to the famous Jones polynomial. The method consisted of solving a set of equations of the Yang-Baxter type. The resulting family of solutions had been first discovered by Akutsu and Wadati [1,2]; their method consisted of extracting representations of B_n from exactly solvable N-state vertex models in statistical mechanics at criticality. Recently our method led to the discovery of a new family of N-state representations of B_n; solutions for N=2, 3 and 4 were given [16,17].

The object of my talk (M.C.) is to give a detailed account of those representations. In section 2 solutions to both families of representations are given up to N=6.

2. N-State Representations of B_n

Artin's braid group B_n [18,19] is generated by a set of (n-1) generators (elementary braids) $g_1, g_2, \ldots, g_{n-1}$ and their inverses subject to the following necessary and sufficient defining relations

$$g_i g_j = g_j g_i \qquad |i-j| \geq 2 \tag{2.1a}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{2.1b}$$

An element β in B_n, called a braid, is a word in the g_i's

$$\beta = g_{i_1}^{\sigma_1} g_{i_2}^{\sigma_2} \ldots, \qquad \sigma_i \in Z$$

Let V be an N-dimensional vector space and $R \in \mathrm{End}(V \otimes V)$ be an $N^2 \times N^2$ matrix that has an inverse. The mapping

$$\rho : B_n \to \mathrm{End}(V^{\otimes n}) \tag{2.2}$$

with

$$\rho(g_i) = I_1 \otimes \ldots \otimes I_{i-1} \otimes R \otimes I_{i+2} \otimes \ldots \otimes I_n \tag{2.3}$$

is a representation of B_n, where $I \in \mathrm{End}(V)$ is the identity matrix, the subscript i means the i^{th} vector space in $V^{\otimes n}$, and R acts on the i^{th} and $(i+1)^{th}$ vector spaces. The form of (2.3) insures that (2.1a) holds. Let us now consider (2.1b). Component wise :