Background-field quantization and the light-cone planar gauge: An exception to Kallosh's theorem

D. G. C. McKeon
Department of Applied Mathematics, The University of Western Ontario, London, Ontario, Canada N6A 5B9

T. N. Sherry
Department of Mathematical Physics, University College, Galway, Ireland

A. M. Chowdhury
Department of Applied Mathematics, The University of Western Ontario, London, Ontario, Canada N6A 5B9

H. C. Lee
Theoretical Physics Branch, Chalk River Nuclear Laboratories, Atomic Energy of Canada Research Company, Chalk River, Ontario, Canada K0J 1P0

M. S. Milgram
Applied Mathematics Branch, Chalk River Nuclear Laboratories, Atomic Energy of Canada Research Company, Chalk River, Ontario, Canada K0J 1P0

(Received 15 August 1986)

The one-loop vacuum-polarization tensor for pure Yang-Mills theory is examined using background-field quantization in the light-cone planar gauge, i.e., with the gauge-fixing Lagrangian $\mathscr{L}_{\rm gf}=(1/2\alpha)n\cdot Q^a D^{2ab}(A)n\cdot Q^b$, where $n^2=0$. The divergent part of the vacuum polarization, $[\Pi_{\mu\nu}(p)]_{\rm div}$, is found to depend on both α and n_μ, and hence is *gauge dependent*. This result does not comply with Kallosh's theorem, according to which the counterterms should be independent of gauge choice. We argue that the occurrence of nonlocal counterterms in the light-cone-type gauges violates one of the implicit assumptions of Kallosh's theorem. We also point out that a similar violation of Kallosh's theorem occurs also in the ordinary light-cone gauge, i.e., using the gauge-fixing Lagrangian $\mathscr{L}_{\rm gf}=-(1/2\alpha)(n\cdot Q)^2$, where $n^2=0$.

The role of the light-cone gauge[1,2] in supersymmetric and superstring models has recently stimulated an examination of radiative processes in Yang-Mills theories quantized in this gauge. One definite advantage of this gauge choice is that it allows one to eliminate unphysical degrees of freedom from the analysis.[3–5]

In earlier studies[2] of this gauge singularities of the form $(k\cdot n)^{-1}$ presented a problem in Feynman integrals when $n^2=0$. The "principal value" prescription, first suggested for handling such singularities[6] in the context of dimensional regularization, proved to be deficient as it led to integrals which do *not* obey naive power counting. Explicit calculations using this prescription also led to results inconsistent with the usual axial anomaly,[7(a)] renormalizability,[7(b)] and the vanishing of the β function in the $N=4$ supersymmetric theory.[4] Subsequently Mandelstam[8] and Leibbrandt[9] found a prescription that overcomes all the above-mentioned deficiencies of the principal-value prescription, is consistent with canonical quantization,[10] and preserves gauge invariance.

Even when it is used in conjunction with the Mandelstam-Leibbrandt prescription, the light-cone gauge still appears peculiar. Specifically, the one-loop vacuum polarization in the Yang-Mills[5,9,11,12] and $N=4$ supersymmetric[9] theories and the vertex functions[5,13,14] in the Yang-Mills theory have nonlocal, unrenormalizable divergences unless field equations are either used to explicitly eliminate the unphysical modes, in which case the unwanted divergences do not appear, or, in case all modes are retained, used to cause the unwanted terms to cancel among themselves in the counter Lagrangian. This feature is not known to exist in any of the other noncovariant gauges that have been studied.

In this paper, for pure Yang-Mills theories, we employ a gauge that is related to the light-cone gauge in the same way that the planar gauge is related to the axial gauge[15]—we call it the light-cone planar gauge. We work within the context of background-field quantization[16] so that the gauge-fixing term in the Lagrangian (analogous to the term used in Ref. 17) is

$$\mathscr{L}_{\rm gf}=\frac{1}{2\alpha}n\cdot Q^a D^2(A)^{ab} n\cdot Q^b \ . \tag{1}$$

Here A_μ^a is the classical background field, Q_μ^b is the quantum field, n_μ is a fixed lightlike vector, and $D_\mu^{ab}(A)=\delta^{ab}\partial_\mu+gf^{apb}A_\mu^p$ is the background covariant derivative. We use the light-cone planar gauge initially, rather than the ordinary light-cone gauge [see Eq. (10)], as we do not lose power-counting arguments when we let α be nonzero. Consequently we can give our results for arbitrary values of α. In the ordinary light-cone gauge such renormalizability arguments force us to restrict our attention to $\alpha=0$.

The gauge-fixing Lagrangian (1) ostensibly satisfies the conditions for Kallosh's theorem: we are dealing with pure Yang-Mills theory, the usual power-counting arguments for renormalizability are still valid and the effective Lagrangian is invariant under the background gauge transformations

$$\delta Q_\mu^a = g f^{abc} Q_\mu^b \theta^c , \tag{2a}$$

$$\delta A_\mu^a = D_\mu^{ab}(A)\theta^b . \tag{2b}$$

Consequently, it is to be expected that the divergence encountered in the two-point function will be independent of the gauge-fixing condition.[18]

In the covariant[19,20] and noncovariant generalized axial[17] gauges, the one-loop two-point function has the divergent piece in 2ω dimensions

$$[\Pi_{\mu\nu}^{ab}(p)]_{\rm div} = \left[\frac{i}{16\pi^2} C_2 \delta^{ab} g^2 \frac{1}{2-\omega}\right] \times \tfrac{11}{3}(p^2\delta_{\mu\nu} - p_\mu p_\nu) . \tag{3}$$

Clearly, Kallosh's theorem is satisfied for these gauges. It is natural to check if the gauge condition of Eq. (1) leads to a divergent contribution to the vacuum polarization consistent with Eq. (3). It is this problem we wish to address now. The Feynman rules appropriate to the calculation of the vacuum polarization can be derived in the usual way. However, it is actually possible to deduce the form of these rules from the analogous Feynman rules for the axial planar gauge.[17] In Ref. 17 the gauge-fixing term in the Lagrangian took the form

$$\mathscr{L}_{\rm gf} = \frac{1}{2\alpha} n\cdot Q^a \frac{D^2(A)^{ab}}{n^2} n\cdot Q^b, \quad n^2 \neq 0 . \tag{4}$$

We see that the parameter redefinition $\alpha n^2 \to \alpha$ gives us a gauge-fixing term of the type we are considering. It is now possible to allow $n^2=0$ without meeting any singularities. For example, the vector propagator obtained after the replacement of αn^2 by α is

$$G_{\mu\nu}^{ab}(p) = \frac{\delta^{ab}}{i(p^2+i\epsilon)}\left[\delta_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n\cdot p} + \frac{p_\mu p_\nu(\alpha+n^2)}{(n\cdot p)^2}\right] . \tag{5a}$$

If we now let $n^2=0$, and using the usual notation $n_\mu^+ \equiv n_\mu$, $p^+ \equiv n_\mu^+ p_\mu$ we obtain

$$G_{\mu\nu}^{ab}(p) = \frac{\delta^{ab}}{i(p^2+i\epsilon)}\left[\delta_{\mu\nu} - \frac{p_\mu n_\nu^+ + p_\nu n_\mu^+}{p^+} + \frac{p_\mu p_\nu}{(p^+)^2}\right] . \tag{5b}$$

This is precisely the propagator which one finds when computing directly in the light-cone planar gauge.

Indeed the Feynman rules for the light-cone planar gauge can now be obtained from those of the axial planar gauge by the two replacements, taken in order,

$$\alpha n^2 \to \alpha, \quad n^2 \to 0 . \tag{6}$$

As in the axial-planar-gauge computation, the only nonzero contributions to the vacuum polarization will be from the Nielsen-Kallosh ghost loop diagram[17] and from the gauge-vector loop diagram. In the evaluation of the associated Feynman integrals we follow the prescription of Refs. 8, 9, and 5 for dealing with the $(n\cdot k)^{-1}$ singularities. (As the principal value prescription of Ref. 6 does not obey power counting, it cannot be expected to yield results consistent with Kallosh's theorem.)

The contribution of the Nielsen-Kallosh ghost loop digram is given by

$$[\Pi_{\mu\nu}^{(1)ab}(p)]_{\rm div} = \left[\frac{i}{16\pi^2} C_2 \delta^{ab} g^2 \frac{1}{2-\omega}\right] \times \tfrac{1}{6}(p^2\delta_{\mu\nu} - p_\mu p_\nu) , \tag{7a}$$

exactly as in the axial-planar-gauge calculation in Ref. 17. This expression is independent of the gauge parameters α and n_μ and is transverse. The vector-loop contribution, although transverse, is dependent on α and n_μ, as follows:

$$\begin{aligned}
[\Pi_{\mu\nu}^{(2)ab}(p)]_{\rm div} = -\left[\frac{i}{16\pi^2} C_2 \delta^{ab} g^2 \frac{1}{2-\omega}\right] \times \Bigg\{ & (p_\mu p_\nu - p^2\delta_{\mu\nu})\left[\frac{7}{2} - 2\alpha\frac{p^-}{p^+} + \alpha^2\frac{p^- p^2}{(p^+)^3}\right] + 2\alpha^2 p_\mu p_\nu \left[\frac{p^-}{p^+}\right]^2 \\
& + (p_\mu n_\nu^+ + p_\nu n_\mu^+)\left[2p^- - 2\alpha\frac{(p^-)^2}{p^+} - \alpha^2\frac{p^2(p^-)^2}{(p^+)^3}\right] \\
& + (p_\mu n_\nu^- + p_\nu n_\mu^-)\left[-2p^+ + 2\alpha p^- - \alpha^2\frac{p^- p^2}{(p^+)^2}\right] \\
& + n_\mu^+ n_\nu^+ \left[-4p^2\frac{p^-}{p^+} + 2\alpha p^2\left[\frac{p^-}{p^+}\right]^2\right] \\
& + (n_\mu^+ n_\nu^- + n_\nu^+ n_\mu^-)\left[2p^2 + \alpha^2\frac{p^4 p^-}{(p^+)^3}\right] - 2\alpha p^2 n_\mu^- n_\nu^- \Bigg\} ,
\end{aligned} \tag{7b}$$

a form which displays quite clearly the nonlocal nature of the divergent part of the vacuum polarization, and hence of the appropriate counterterms. The combined result can be more compactly expressed in terms of four tensors transverse to p_μ:

$$P_{\mu\nu}=p_\mu p_\mu - p^2\delta_{\mu\nu}\ , \tag{8a}$$

$$R_{\mu\nu}=r_\mu r_\nu,\ \ r_\mu \equiv p^+ p_\mu^- - zp_\mu/2,\ \ z\equiv 2p^+p^-/p^2\ , \tag{8b}$$

$$S_{\mu\nu}=r_\mu s_\nu + r_\nu s_\mu,\ \ s_\mu \equiv p^- n_\mu^+ - zp_\mu/2\ , \tag{8c}$$

$$T_{\mu\nu}=s_\mu s_\nu\ , \tag{8d}$$

of which the last three are gauge dependent and nonlocal through their dependence on the two vectors r_μ and s_μ and the variable u.

The total vacuum polarization is then

$$\begin{aligned}[\Pi^{ab}_{\mu\nu}(p)]_{\mathrm{div}} = & \left[\frac{i}{16\pi^2}C_2\delta^{ab}g^2\frac{1}{2-\omega}\right] \\ & \times\left[(-\tfrac{11}{3}+z\alpha'-\tfrac{1}{2}z\alpha'^2)P_{\mu\nu}\right. \\ & +2\alpha' R_{\mu\nu}-\left[\frac{4}{z}+\alpha'^2\right]S_{\mu\nu} \\ & \left.+\left[\frac{2}{z}-\alpha'\right]T_{\mu\nu}\right]\ ,\end{aligned} \tag{9}$$

where $\alpha'\equiv \alpha p^2/(p^+)^2$, and in which the transversality, gauge dependence, and nonlocality of the vacuum polarization is clearly displayed.

From Eq. (7) or (9) it is obvious that polarization in the light-cone planar gauge *does not* take the simple form of Eq. (3) which we might have expected on the basis of Kallosh's theorem. It follows that the counterterms required to cancel the divergent point of the vacuum-polarization tensor are gauge dependent in the light-cone planar gauge—even in background-field quantization. This result is clearly in violation of Kallosh's theorem.

A similar violation of Kallosh's theorem occurs also in the ordinary light-cone gauge in background-field quantization. This gauge corresponds to using the gauge-fixing term

$$\mathscr{L}_{\mathrm{gf}} = -\frac{1}{2\alpha}(n\cdot Q^a)^2,\ \ n^2=0\ , \tag{10}$$

in the limit $\alpha\to 0$ (to avoid losing power-counting arguments). It turns out that in this case, just as occurred in the axial ($n^2\neq 0$) cases, the computation of the vacuum-polarization tensor is exactly the same as using conventional techniques.[5,9,11,12] The result is again gauge dependent and nonlocal,

$$\begin{aligned}[\Pi^{ab}_{\mu\nu}(p)]_{\mathrm{div}} = & \left[\frac{i}{16\pi^2}C_2\delta^{ab}g^2\frac{1}{2-\omega}\right] \\ & \times\left[-\tfrac{11}{3}P_{\mu\nu}-\frac{4}{z}S_{\mu\nu}+\frac{2}{z}T_{\mu\nu}\right]\ ,\end{aligned} \tag{11}$$

and indeed coincides with Eq. (9) in the limit $\alpha\to 0$.

It is interesting to contrast this result with the results obtained in the axial gauges, i.e., using the gauge-fixing condition (10) but for $n^2\neq 0$. In that case,[17] the gauges for $\alpha\neq 0$ were pathological—Kallosh's theorem was not applicable for $\alpha\neq 0$ due to the $O(1)$ behavior of the vector propagator at large momenta. However, in the limit $\alpha\to 0$ the correct $O(p^{-2})$ behavior was obtained and Kallosh's theorem was satisfied. In the light-cone gauge case, however, as demonstrated by Eq. (11), Kallosh's theorem is not satisfied even in the limit $\alpha\to 0$.

It is important to try to understand why Kallosh's theorem is being violated in these calculations, as this theorem is given as one of the reasons for using background-field quantization. The *explicit assumptions* of Kallosh's theorem are (1) pure Yang-Mills gauge theory and (2) background gauge invariance maintained by the quantization procedure. It is quite clear that both of these assumptions are still valid. However, on closer examination of Kallosh's theorem, we notice two *implicit assumptions*: namely, (3) power-counting arguments are valid, and (4) counterterms are *local,* i.e., the divergences of the theory have *local* structure. Assumption (3) has previously been noted[17] in the context of the axial gauges. Assumption (4) has not been highlighted before, to our knowledge, as it is only recently that the phenomenon of nonlocal counterterms has been noticed.[9,11,12] In the light-cone and light-cone planar gauges assumption (4) is not valid and, consequently, we have no reason to expect that Kallosh's theorem should be applicable to these gauges.

We wish to stress that quantization in the background-field formalism involves a gauge-fixing condition only on the quantum fields and not on the external background fields. Consequently, the program for eliminating non-physical modes advocated in Ref. 5 is not immediately applicable.

The breakdown of Kallosh's theorem due to the presence of gauge-dependent and nonlocal infinite terms in the vacuum polarization once again raises the question whether Yang-Mills theory in the light-cone (planar) gauge is renormalizable, this time in background-field quantization. In conventional quantization, where a similar situation was encountered,[5,9,11,12,14] renormalization was achieved by using a crucial field equation in the total counter Lagrangian to affect a complete cancellation of all the unwanted counterterms;[5] the counter Lagrangian is constructed from all the radiatively corrected primitive Green's functions. Because of the presence of the background field as well as the quantum field, the situation in background-field quantization is more complicated. For example, the construction of the part of the counter Lagrangian quadratic in the vector fields calls for the radiative-corrected vacuum polarizations $\langle Q^a_\mu Q^b_\nu\rangle$ and $\langle Q^a_\mu A^b_\nu\rangle$ as well as the quantity $\langle A^a_\mu A^b_\nu\rangle$ calculated here. Furthermore, because it is non-Abelian it always assures that the field equation relates counterterms appearing in Green's functions with numbers of external points, the expected cancellation cannot be demonstrated until at least the various one-loop, three-point functions are also calculated. That task is considerably beyond the scope of this work.

In this paper we have shown, by direct computation in the light-cone planar gauge using background-field quantization, that the divergent part of the one-loop vacuum-polarization tensor is highly gauge dependent. We have also noted that a similar result holds in the ordinary light-cone ($\alpha=0$) gauge. Both of these results appear to contradict Kallosh's theorem. However, we have argued that the occurrence of nonlocal counterterms in both of these gauges violates one of the implicit assumptions built into the proof of Kallosh's result. Hence it is not surprising that Kallosh's theorem is not satisfied in explicit computations.

We thank the Natural Sciences and Engineering Research Council, Canada, and the University of Western Ontario Academic Development Fund for generous support. One of the authors (T.N.S.) also thanks the Department of Applied Mathematics at the University of Western Ontario for their hospitality and support during his visit.

---

[1]S. Weinberg, Phys. Rev. **150**, 1313 (1966).

[2]E. Tomboulis, Phys. Rev. D **8**, 2736 (1973); J. M. Cornwall, *ibid.* **10**, 500 (1974); A. Casher, *ibid.* **14**, 452 (1976); C. Thorn, *ibid.* **20**, 1934 (1979).

[3]J. Bjorken, J. Kogut, and D. Soper, Phys. Rev. D **1**, 2901 (1970); J. Scherk and J. H. Schwarz, Gen. Relativ. Gravit. **6**, 537 (1975).

[4]D. Capper, J. Dulwich, and M. Litvak, Nucl. Phys. **B241**, 463 (1984).

[5]H. C. Lee and M. Milgram, Phys. Rev. Lett. **55**, 2122 (1985); Nucl. Phys. **B268**, 543 (1986).

[6]W. Kummer, Acta Phys. Austriaca **41**, 315 (1975).

[7](a) D. Capper and M. Litvak, J. Phys. A **18**, 955 (1985); (b) H. C. Lee and M. S. Milgram, Z. Phys. C **28**, 579 (1985).

[8]S. Mandelstam, Nucl. Phys. **B213**, 1493 (1983).

[9]G. Leibbrandt, Phys. Rev. D **29**, 1699 (1984).

[10]A. Bassetto, M. Dalbosco, I. Lazzizzera, and R. Soldati, Phys. Rev. D **31**, 2012 (1985).

[11]G. Leibbrandt and S. Nyeo, Phys. Lett. **140B**, 417 (1984).

[12]A. Andrasi, G. Leibbrandt, and S. Nyeo, University of Guelph Report No. 1985-100 (unpublished).

[13]G. Leibbrandt and T. Matsuki, Phys. Rev. D **31**, 934 (1985).

[14]A. Bassetto, M. Dalbosco, and R. Soldati, Phys. Lett. **159B**, 311 (1985).

[15]D. Capper and G. Leibbrandt, Phys. Rev. D **25**, 1002 (1982).

[16]B. DeWitt, Phys. Rev. **162**, 1195 (1967).

[17]G. McKeon, S. Phillips, S. Samant, T. Sherry, H. C. Lee, and M. Milgram, Phys. Lett. **161B**, 319 (1985).

[18]R. Kallosh, Nucl. Phys. **B78**, 293 (1974).

[19]L. Abbott, Nucl. Phys. **B185**, 189 (1981).

[20]D. Capper and A. MacLean, Nucl. Phys. **B203**, 413 (1981).