Gauge invariance of the one-loop effective potential in $M^d \times S^1$ Kaluza-Klein theory

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By adjusting the Weyl factor in $(d+1)$-dimensional Kaluza-Klein theory, we obtain a one-parameter family of gauges which give rise to propagating, massive ghosts in the compactified theory on $M^d \times S^1$. We calculate the one-loop Casimir energy of this vacuum for all values of the gauge parameter S and arbitrary noncompact "space-time" dimension d. As expected, the result is independent of S and coincides with those previously obtained in other gauges. Our calculation confirms recent formal proofs due to Cohler and Chodos and Yasuda concerning the gauge independence of the graviton one-loop effective potential.

I. INTRODUCTION

Kaluza-Klein theories[1,2] provide a possible geometrical unification of gravity with the other fundamental interactions. Moreover, the existence of extra space-time dimensions arises naturally in formulations of extended supergravity[3] and is an essential feature of consistent string theories.[4] Most recent calculations are either purely classical, or deal with one-loop quantum corrections to the classical vacuum energy. Although the classical analyses lead to considerable insight into the geometrical structure of such theories, their direct physical significance is weakened by the fact that the extra dimensions are expected *a priori* to "curl up" to sizes of the order of the Planck scale ($\sim 10^{-33}$ cm) in order for the effective gauge coupling constants to have reasonable values.

A potentially more fruitful approach taken recently by many authors[5–9] is to calculate the one-loop effective potential due to either matter fields[5] or the graviton itself.[6–9] In particular, by finding self-consistent solutions to the quantum-corrected equations of motion for the background field, it is possible to predict the effective coupling constants and radii of the internal dimensions.[5,8,9] However, as pointed out in Ref. 8, the graviton one-loop effective potential is not, in general, gauge invariant, although its minima are expected to be so. Two recent papers[10,11] have provided formal proofs of the gauge invariance of the graviton one-loop effective potential for certain classes of gauge choices when the background is chosen to be a solution to the classical equations of motion. Unfortunately, in order to predict coupling constants self-consistently in quantized Kaluza-Klein theory, it is necessary to first evaluate the effective potential off-shell and then fine-tune parameters (such as the cosmological constant) in order to ensure that the chosen background minimizes the quantum-corrected effective action. The hope is that the predicted physical parameters emerging from this manifestly noncovariant procedure will nonetheless be independent of the gauge choice. Recent work,[12] however, seems to indicate that this hope may not be realized in practice.

We are thus motivated to analyze in some detail the calculation of the Casimir energy of the gravitational field in $D=d+1$ space-time dimensions compactified to $M^d \times S^1$. This calculation was first done for $d=4$ in a specific gauge by Applequist and Chodos.[6] In addition to generalizing the number of space-time dimensions to d, which does not in itself present any severe complications, we introduce an arbitrary Weyl or conformal factor into the parametrization of the $(d+1)$-metric in terms of d-dimensional fields. Although the resulting field redefinition does not affect the classical dynamics of the reduced d-dimensional theory, it does provide an interesting one-parameter family of gauge choices with nontrivial quantum properties. Specifically, propagating massive ghosts, scalars as well as graviton modes in the reduced theory all contribute to the one-loop effective potential in this family of gauges. We are, therefore, able to verify with explicit calculations the formal proofs[10,11] of the gauge independence of the effective potential and Casimir energy.

This paper is organized as follows. In Sec. II we briefly review the $(d+1)$-dimensional Kaluza-Klein theory with a Weyl factor. We derive the effective d-dimensional linearized action and discuss its dynamics. Section III reviews the calculation of the effective potential in the cylindrical gauge[6] generalized to $d+1$ dimensions, while Sec. IV contains the calculation in the new family of gauges. In Sec. V we close with conclusions and comments on future work.

II. LINEARIZED KALUZA-KLEIN ACTION WITH ARBITRARY WEYL FACTOR

We consider a Kaluza-Klein theory in $D=d+1$ dimensions with metric parametrized as follows:

$$\bar{g}_{AB}=\left[\frac{\phi}{\phi_0}\right]^S\begin{bmatrix} g_{\mu\nu}+\phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{bmatrix} . \tag{2.1}$$

(In our notation, all quantities defined in the full D-dimensional space-time are barred and carry upper-case latin indices $\{A,B,\ldots=0,1,\ldots,d-1,D\}$. Quantities defined on the d-dimensional reduced space-time carry greek indices $\{\mu,\nu,\ldots=0,1,\ldots,d-1\}$. d-dimensional space-time coordinates will be labeled by x^μ, while the remaining coordinate y is periodic with period $2\pi R$.)

As yet no assumptions have been made about the coordinate dependence of $\bar{g}_{AB}=\bar{g}_{AB}(x,y)$. The so-called Weyl factor or conformal factor $(\phi/\phi_0)^S$ has been normalized by a constant ϕ_0 which will be chosen as the vacuum expectation value of ϕ, for reasons that will become clear in the subsequent calculations. Note that a change in the Weyl exponent S induces a field redefinition of the effective d-dimensional fields $g_{\mu\nu}$, A_μ, and ϕ without changing the dynamical content of the theory. In particular, there appears to be an ambiguity in the definition of the reduced space-time metric $g_{\mu\nu}$. This ambiguity can only be resolved by determining the coupling of matter fields to geometrical quantities. For example, if one considers test particles in $d+1$ dimensions which traverse geodesics of $\bar{g}_{AB}$, then there exists a natural choice of Weyl exponent, namely, $S=0$, for which these geodesics project onto geodesics of $g_{\mu\nu}$.[13,14]

The action is the ordinary Einstein-Hilbert action in D dimensions:

$$\bar{I}=\frac{1}{16\pi\bar{G}}\int d^Dx\sqrt{-\bar{g}}\;\bar{R}_{AB}\bar{g}^{AB} , \tag{2.2}$$

where

$$\bar{R}_{AB}=\bar{\Gamma}^M_{AB,M}-\bar{\Gamma}^M_{AM,B}-\bar{\Gamma}^M_{AB}\bar{\Gamma}^N_{MN}+\bar{\Gamma}^M_{AN}\bar{\Gamma}^N_{MB} \tag{2.3}$$

and

$$\bar{\Gamma}^M_{AB}\equiv\tfrac{1}{2}\bar{g}^{MN}(\bar{g}_{AN,B}+\bar{g}_{BN,A}-\bar{g}_{AB,N}) . \tag{2.4}$$

Instead of writing out the full action (2.2) in terms of the fields $g_{\mu\nu}$, A_μ, and ϕ, we perform the expansion

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} , \tag{2.5}$$

$$\phi=\phi_0(1+\psi) , \tag{2.6}$$

$$A_\mu=0+A_\mu , \tag{2.7}$$

where $h_{\mu\nu}$, ψ, and A_μ are small fluctuations of $O(\epsilon)$ which depend on all coordinates $\{x^\mu,y\}$. In terms of the full D-dimensional metric, this is equivalent to setting

$$\bar{g}_{AB}=\bar{g}^{\,0}_{AB}+\bar{h}_{AB} , \tag{2.8}$$

where

$$\bar{g}^{\,0}_{AB}=\begin{bmatrix}\eta_{\mu\nu} & 0\\ 0 & \phi_0\end{bmatrix} \tag{2.9}$$

and to $O(\epsilon)$

$$\bar{h}_{AB}=\begin{bmatrix} h_{\mu\nu}+S\psi\eta_{\mu\nu} & \phi_0 A_\mu \\ \phi_0 A_\nu & (S+1)\phi_0\psi\end{bmatrix} . \tag{2.10}$$

The reason for our normalization of the Weyl factor by ϕ_0 now becomes apparent: it ensures that the expansion in Eqs. (2.5)–(2.7) yields the same D-dimensional background $\bar{g}^{\,0}_{AB}$ for all choices of Weyl factor, thus avoiding the rescaling of the coordinates which was required by Applequist and Chodos[6] in order to yield a Minkowski space-time background.

It is now straightforward to calculate the action to second order in the fluctuations. The result is (note that $\bar{I}^{(0)}=\bar{I}^{(1)}=0$ because our background is a flat solution to the D-dimensional field equations)

$$\bar{I}^{(2)}=\frac{{\phi_0}^{1/2}}{16\pi\bar{G}}\int d^dx[\mathcal{L}_1(h^2)+\mathcal{L}_2+\mathcal{L}_3(\psi^2)+\mathcal{L}_4(\psi,h)+\mathcal{L}_5(A,\psi,h)] \tag{2.11}$$

with

$$\mathcal{L}_1(h^2)=\tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{1}{4}h\Box h+\tfrac{1}{2}hh''-\tfrac{1}{2}h^{\mu\nu}h^{\kappa}{}_{\mu,\kappa,\nu}-\frac{1}{4\phi_0}(h\partial_y{}^2h-h_{\mu\nu}\partial_y{}^2h^{\mu\nu}) , \tag{2.12a}$$

$$\mathcal{L}_2=\mathcal{L}_{\rm EM}=-\tfrac{1}{4}\phi_0F_{\mu\nu}F^{\mu\nu} , \tag{2.12b}$$

$$\mathcal{L}_3(\psi^2)=a_1\psi\Box\psi-\frac{a_2}{\phi_0}\psi\partial_y{}^2\psi , \tag{2.12c}$$

$$\mathcal{L}_4(\psi,h)=a_3\psi(h''-\Box h)-\frac{a_4}{\phi_0}\psi\partial_y{}^2h , \tag{2.12d}$$

$$\mathcal{L}_5(A,\psi,h)=2a_4\psi\partial_y(\partial_\mu A^\mu)+h\partial_y(\partial_\mu A^\mu)-h^{\mu\nu}\partial_y(A_{\mu,\nu}) , \tag{2.12e}$$

where $\Box$ denotes the d-dimensional Laplacian $\partial^\mu\partial_\mu$,

$$h\equiv h_{\mu\nu}\eta^{\mu\nu} , \tag{2.13a}$$

$$h''\equiv h^{\mu\nu}{}_{,\nu,\mu} , \tag{2.13b}$$

and

$$a_1\equiv -S(d-1)(Sd+2)/4 , \tag{2.14a}$$

$$a_2\equiv S^2d(d-1)/4 , \tag{2.14b}$$

$$a_3\equiv[S(d-1)+1]/2 , \tag{2.14c}$$

$$a_4\equiv S(d-1)/2 . \tag{2.14d}$$

Consider the harmonic expansion of an arbitrary function $f(x^\mu,y)$:

$$f(x^\mu,y)=\sum_{n=-\infty}^{\infty}f^{(n)}(x^\mu)e^{iny/R} . \tag{2.15}$$

In the present analysis all fields are real, so that $f^{(n)}=f^{*(-n)}$ and

$$\int dy\, f(x^\mu,y)g(x^\mu,y)=2\pi R \sum_{n=-\infty}^{\infty} f^{(n)}(x^\mu)g^{*(n)}(x^\mu)$$
$$=2\pi R \sum_{n=-\infty}^{\infty} \tilde{f}^{(n)}(x^\mu)\tilde{g}^{(n)}(x^\mu)\ . \tag{2.16}$$

In Eq. (2.16) we have defined the real fields:

$$\tilde{f}^{(n)}(x^\mu)\equiv\sqrt{2}\,\mathrm{Re}f^{(n)}(x^\mu),\quad n>0\ , \tag{2.17a}$$

$$\tilde{f}^{(n)}(x^\mu)\equiv\sqrt{2}\,\mathrm{Im}f^{(n)}(x^\mu),\quad n<0\ , \tag{2.17b}$$

$$\tilde{f}^{(0)}(x^\mu)\equiv f^{(0)}(x^\mu)\ . \tag{2.17c}$$

Similarly,

$$\int dy\, f(x^\mu,y)\frac{\partial}{\partial y}g(x^\mu,y)=2\pi R \sum_{n=-\infty}^{\infty} \frac{n}{R}\tilde{f}^{(-n)}\tilde{g}^{(n)} \tag{2.18}$$

and

$$\int dy\, f(x^\mu,y)\frac{\partial^2}{\partial y^2}g(x^\mu,y)$$
$$=-2\pi R \sum_{n=-\infty}^{\infty} \frac{n^2}{R^2}\tilde{f}^{(n)}(x^\mu)\tilde{g}^{(n)}(x^\mu)\ . \tag{2.19}$$

In terms of these new expansion coefficients, quantities of the form in Eq. (2.18) give rise to mixing between different n sectors. In particular, only $\mathcal{L}_5(A,\psi,h)$ in Eq. (2.11) contains such terms. Such mixing is, however, eliminated by the redefinition $\tilde{A}^{(n)}\rightarrow\tilde{A}^{(-n)}$.

Using Eqs. (2.15)—(2.18) in Eq. (2.11) we get

$$\bar{I}^{(2)}=\frac{1}{16\pi G_d}\sum_{n=-\infty}^{\infty}\int d^4x(\mathcal{L}_1^{(n)}+\mathcal{L}_2^{(n)}+\mathcal{L}_3^{(n)}$$
$$+\mathcal{L}_4^{(n)}+\mathcal{L}_5^{(n)})\ , \tag{2.20}$$

where

$$\mathcal{L}_1^{(n)}\equiv\tfrac{1}{4}(h_{\mu\nu}^{(n)}\Box h^{(n)\mu\nu}-h^{(n)}\Box h^{(n)})$$
$$+\tfrac{1}{2}(h^{(n)}h^{(n)\prime\prime}-h^{(n)\mu\nu}h^{(n)\kappa}{}_{\mu,\kappa,\nu})$$
$$+\frac{m_n{}^2}{4}[(h^{(n)})^2-h_{\mu\nu}^{(n)}h^{(n)\mu\nu}]\ , \tag{2.21a}$$

$$\mathcal{L}_2^{(n)}=-\tfrac{1}{4}F_{\mu\nu}^{(n)}F^{(n)\mu\nu}\ , \tag{2.21b}$$

$$\mathcal{L}_3^{(n)}=a_1\psi^{(n)}\Box\psi^{(n)}+a_2m_n{}^2\psi^{(n)^2}\ , \tag{2.21c}$$

$$\mathcal{L}_4^{(n)}=a_3\psi^{(n)}(h^{(n)\prime\prime}-\Box h^{(n)})+a_4m_n{}^2\psi^{(n)}h^{(n)}\ , \tag{2.21d}$$

$$\mathcal{L}_5^{(n)}=m_n(-2a_4\psi^{(n)}\partial_\mu A^{(n)\mu}-h^{(n)}\partial_\mu A^{(n)\mu}$$
$$+h^{(n)\mu\nu}\partial_\nu A_\mu^{(n)})\ . \tag{2.21e}$$

In the above the tildes over the fields have been dropped (they will henceforth be implicit); the vector potential has been rescaled, $A_\mu\rightarrow A_\mu/\sqrt{\phi_0}$; the "mass" appearing in each sector is defined by $m_n=n/(R\sqrt{\phi_0})$; and the effective d-dimensional gravitational constant G_d is defined by

$$G_d\equiv\left[\frac{\bar{G}}{2\pi R\sqrt{\phi_0}}\right]\ . \tag{2.22}$$

Note that only the coordinate-invariant circumference of the compact direction $\int\sqrt{g_{DD}}\,dy=2\pi R\sqrt{\phi_0}\equiv2\pi\bar{R}$ appears in these expressions. The final answer, therefore, will also be invariant under rescalings of the y coordinate. We will for simplicity, henceforth, drop the ϕ_0 from our expression, and R will be understood to refer to the invariant length $\bar{R}$.

A few brief comments concerning the dynamical content of the reduced action $\bar{I}^{(2)}$ are in order here. (See also Refs. 15—17). The full D-dimensional gravity theory contains $\frac{1}{2}D(D+1)$ functions of the D coordinates. Of these only $\frac{1}{2}D(D-3)$ propagate dynamically due to the manifest coordinate invariance in the theory. The reduced action $\bar{I}^{(2)}$ has the same dynamical content but in a drastically altered form. The harmonic expansion has yielded a countable infinity of propagating modes each of which is now a function of only d coordinates x^μ. Note that the different choices of Weyl exponent can radically alter the form of $\bar{I}^{(2)}$ without changing its dynamical content. For example, if $S=-(d-1)^{-1}$ (i.e., $a_3=0$) then there is no derivative coupling between $h_{\mu\nu}^{(n)}$ and $\psi^{(n)}$, while if $S=0$ then the kinetic term for $\psi^{(n)}$ vanishes. The equivalence of the different forms of $\bar{I}^{(2)}$ can be verified by observing that one can change the Weyl exponent in $\bar{I}^{(2)}$ from S to S' by performing the following invertible field redefinitions:

$$h_{\mu\nu}^{(n)}\rightarrow h_{\mu\nu}^{(n)}+\frac{(S'-S)}{(S+1)}\psi^{(n)}\eta_{\mu\nu}\ , \tag{2.23a}$$

$$\psi^{(n)}\rightarrow\frac{(S'+1)}{(S+1)}\psi^{(n)}\ , \tag{2.23b}$$

$$A_\mu^{(n)}\rightarrow A_\mu^{(n)}\ . \tag{2.23c}$$

The $n=0$ sector contains $d(d-3)/2$ massless graviton modes, $(d-2)$ gauge vector modes, and a scalar mode, for a total of $D(D-3)/2$ as required. This sector is also invariant under space-time coordinate transformation and has an (Abelian) gauge symmetry. These are the only unbroken gauge symmetries in the reduced theory,[16,17] and correspond to the generators of y-independent coordinate transformations in the full theory.

The Lagrangian in the nth Fourier sector ($n\neq0$) is invariant under the following gauge transformations:

$$h_{\mu\nu}^{(n)}\rightarrow h_{\mu\nu}^{(n)}+\frac{1}{m_n}(\rho_{\mu,\nu}^{(n)}+\rho_{\nu,\mu}^{(n)})-S\eta_{\mu\nu}\lambda^{(n)}$$
$$+\frac{(S+1)}{m_n{}^2}\lambda_{,\mu\nu}^{(n)}\ , \tag{2.24a}$$

$$A_\mu^{(n)}\rightarrow A_\mu^{(n)}+\rho_\mu^{(n)}\ , \tag{2.24b}$$

$$\psi^{(n)}\rightarrow\psi^{(n)}+\lambda^{(n)}\ , \tag{2.24c}$$

where $\rho_\mu^{(n)}$ is an arbitrary vector field and $\lambda^{(n)}$ an arbitrary scalar. Thus, the fields $A_\mu^{(n)}$ and $\psi^{(n)}$, $n\neq0$, are pure gauge, and can be transformed to zero without loss of dynamical content. In this gauge (which is the analogue of the unitary gauge in the spontaneously broken Abelian

Higgs model) the physical content of the massive sectors is manifest. Each $n\neq 0$ sector contains a d-dimensional massive "graviton" with $(d+1)(d-2)/2=D(D-3)/2$ dynamical modes. In the language of spontaneously broken symmetry, the generators of coordinate transformations in these sectors correspond to symmetries which are spontaneously broken by the chosen vacuum solution, $\bar{g}^{\,0}_{AB}$; the massless vector and scalar fields are the Goldstone bosons which get absorbed to produce the massive graviton. As we shall see in the next sections, only the massive modes contribute to the one-loop effective potential.

III. EFFECTIVE POTENTIAL IN CYLINDRICAL GAUGE

The one-loop effective action can be formally expressed as follows:[18]

$$\Gamma[\phi_i^0]=\bar{I}[\phi_i^0]+\tfrac{1}{2}\ln\det\mathscr{D}^{-1}{}_{ij}\ , \tag{3.1}$$

where ϕ_i^0 generically denotes the background fields $\phi_i^0\equiv[g^0_{\mu\nu},A^0_\mu,\phi^0]=[\eta_{\mu\nu},0,\phi_0]$, the inverse propagator $\mathscr{D}^{-1}$ is defined by

$$\tfrac{1}{2}\int d^Dx\, h_i\mathscr{D}^{-1}{}_{ij}h_j=\bar{I}^{(2)} \tag{3.2}$$

and h_i denotes the fluctuations $[h_{\mu\nu},A_\mu,\psi]$. Expression (3.1) is not well defined, however, because the general covariance of the original action manifests itself as a gauge invariance of $I^{(2)}$ under the field transformations:

$$\bar{h}_{AB}\to\bar{h}_{AB}+\Sigma_{A,B}+\Sigma_{B,A} \tag{3.3}$$

or, in terms of the d-dimensional fields:

$$h_{\mu\nu}\to h_{\mu\nu}+(\Sigma_{\mu,\nu}+\Sigma_{\nu,\mu})-\frac{2S}{S+1}\eta_{\mu\nu}\frac{\Sigma_{D,D}}{\phi_0}\ , \tag{3.4}$$

$$A_\mu\to A_\mu+\frac{1}{\phi_0}(\Sigma_{\mu,D}+\Sigma_{D,\mu})\ , \tag{3.5}$$

$$\psi\to\psi+\frac{2}{(S+1)}\Sigma_{D,D}\ . \tag{3.6}$$

As is well known, this invariance gives rise to a singularity in the functional integral, which must be removed by the addition to the Lagrangian of a gauge-fixing term and a corresponding Faddeev-Popov[19] term. The one-loop effective action thus modified is

$$\Gamma[\phi_i^0]=\bar{I}[\phi_i^0]+\tfrac{1}{2}\ln\det\overline{\mathscr{D}}^{\,-1}{}_{ij}-\ln\det\Delta_{\rm FP}\ , \tag{3.7}$$

where $\det\Delta_{\rm FP}$ is the Faddeev-Popov determinant and instead of (3.2) now

$$\tfrac{1}{2}\int d^Dx\, h_i(x)\overline{\mathscr{D}}^{\,-1}{}_{ij}(x)h_j(x)=\bar{I}^{(2)}+\bar{I}^{(2)}_{\rm gf}\ . \tag{3.8}$$

The cylindrical gauge is defined by the following gauge-fixing condition:

$$\partial_y\bar{g}^{\,AD}=0\ . \tag{3.9}$$

The calculation in this gauge was first done by Appelquist and Chodos[6] for $d=4$ and $S=0$. Since it is much simpler to calculate in Euclidean space than in Minkowski space, we Wick rotate as follows:

$$\left[t,i\bar{I},\int d^dx,\int d^dk\right]_{\rm Minkowski}\to\left[iX_0,-\bar{I},i\int d^dx,-i\int d^dk\right]_{\rm Euclidean}\ .$$

We also transform to d-dimensional momentum space to obtain

$$V^{(1)}_{\rm eff}=\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}\ln\det\overline{\mathscr{D}}^{\,-1}{}_{ij}(k)-\ln\det\Delta_{\rm FP}\ , \tag{3.10}$$

where

$$\overline{\mathscr{D}}^{\,-1}{}_{ij}(k)\equiv\int d^dx\, e^{ikx}\overline{\mathscr{D}}^{\,-1}{}_{ij}(x)\ , \tag{3.11}$$

with $\overline{\mathscr{D}}^{\,-1}{}_{ij}(x)$ defined by Eq. (3.8). In Eq. (3.10), $V^{(1)}_{\rm eff}$ is expressed as energy per unit $(d-1)$-volume. In order to express it in terms of d spatial dimensions it is necessary to divide by $(2\pi R)$.

The relevant gauge-fixing term that must be added to the action following procedures described in the last section can be reduced to

$$\int d^dx\,dy\,\mathscr{L}_{\rm gf}=\int\frac{1}{2\alpha}(\partial_y\bar{g}^{\,AD})^2d^dx\,dy=\frac{2\pi R}{2\alpha}\sum_n\int d^dx(m_n{}^2)[(A^{(n)}_\mu)^2+(\psi^{(n)})^2]\ . \tag{3.12}$$

This gauge constraint is trivially implementable in the limit $\alpha\to 0$, because it merely introduces a product of δ functions $\prod_{n\neq 0}\delta(A^{(n)}_\mu(x))\delta(\psi^{(n)}(x))$ into the functional integral. Performing the integration then removes all expressions containing $A^{(n)}_\mu$ and $\psi^{(n)}$ $(n\neq 0)$ from the second-order Lagrangian. Note that we have not fixed the residual D gauge degrees of freedom in the $n=0$ sector, but this is not important since only the massive sectors contribute to the one-loop effective potential. Note also that in this gauge the Weyl exponent S has completely disappeared from the calculation. The remaining inverse propagator for each n is simply that of a massive spin-2 particle with mass $m_n\sim n/R$.

We therefore have the following expression:

$$\bar{I}^{(2)}+\bar{I}^{(2)}_{\rm gf}=\frac{2\pi R}{2}\sum_n\int\frac{d^dk}{(2\pi)^d}h^{(n)}_{\mu\nu}(k)\overline{\mathscr{D}}^{\,(n)-1}_{\mu\nu,\lambda\sigma}(k)h^{(n)}_{\lambda\sigma}(k)\ , \tag{3.13}$$

where

$$\begin{aligned}\overline{\mathscr{D}}^{\,(n)-1}_{\mu\nu,\lambda\sigma}(k)&=\tfrac{1}{2}(m_n{}^2+k^2)[\delta_{\mu\nu}\delta_{\lambda\sigma}-\tfrac{1}{2}(\delta_{\lambda\mu}\delta_{\sigma\nu}+\delta_{\lambda\nu}\delta_{\sigma\mu})]\\&\quad+\tfrac{1}{4}(\delta_{\mu\lambda}k_\sigma k_\nu+\delta_{\nu\lambda}k_\sigma k_\mu+\delta_{\mu\sigma}k_\lambda k_\nu+\delta_{\nu\sigma}k_\lambda k_\mu)\\&\quad-\tfrac{1}{2}(\delta_{\mu\nu}k_\lambda k_\sigma+\delta_{\lambda\sigma}k_\mu k_\nu)\end{aligned} \tag{3.14}$$

and we have suppressed the coupling constant for simplicity.

In order to evaluate the determinant, we expand eigenvectors $X^{(n)}_{\mu\nu}$ of the inverse propagator $\overline{\mathscr{D}}^{(n)-1}_{\mu\nu,\lambda\sigma}$ in a complete orthonormal basis as follows:

$$X^{(n)}_{\mu\nu}=a\hat{k}_\mu\hat{k}_\nu+b\sum_{i=1}^{d-1}p^i_\mu p^i_\nu+\sum_{i=1}^{d-1}b'_i p^i_\mu p^i_\nu+\sum_i C_i(p^i_\mu\hat{k}_\nu+p^i_\nu\hat{k}_\mu)+\sum_{i\neq j}d_{ij}(p^i_\mu p^j_\nu+p^i_\nu p^j_\mu)\ , \tag{3.15}$$

where $\hat{k}_\mu\equiv k_\mu/|k|$ and p^i_μ, $i=1$ to $d-1$ satisfy

$$\hat{k}_\mu p^i_\nu\delta^{\mu\nu}=0,\ \ p^i_\mu p^j_\nu\delta^{\mu\nu}=\delta^{ij}\ , \tag{3.16}$$

$$\delta_{\mu\nu}=\hat{k}_\mu\hat{k}_\nu+\sum_i p^i_\mu p^i_\nu\ , \tag{3.17}$$

and we require $\sum_i b'_i=0$. In this basis $\mathscr{D}^{-1}{}_{\mu\nu,\lambda\sigma}$ has the following simple representation:

$$\mathscr{D}^{(n)-1}=\begin{bmatrix}\mathscr{D}_1^{(n)-1} & 0\\ 0 & \mathscr{D}_2^{(n)-1}\end{bmatrix}\ , \tag{3.18}$$

where

$$\mathscr{D}_1^{(n)-1}=\begin{bmatrix}0 & \frac{(d-1)}{2}m_n{}^2\\ \frac{1}{2}m_n{}^2 & \frac{(d-2)}{2}(m_n{}^2+k^2)\end{bmatrix} \tag{3.19}$$

and

$$\mathscr{D}_2^{(n)-1}=\begin{bmatrix}-\frac{1}{2}(k^2+m_n{}^2)I_1 & & \\ & -\frac{1}{2}m_n{}^2I_2 & \\ & & -\frac{1}{2}(k^2+m_n{}^2)I_3\end{bmatrix}\ . \tag{3.20}$$

In Eq. (3.20), I_1 is the $(d-2)$-dimensional unit matrix acting on tensors $\sum b'_i p^i_\mu p^i_\nu$, I_2 is the $(d-1)$-dimensional unit matrix acting on $\sum_i C_i(p^i_\mu\hat{k}_\nu+p^i_\nu\hat{k}_\mu)$ and I_3 is the $(d-1)(d-2)/2$-dimensional unit matrix acting on $\sum_{i\neq j}d_{ij}(p^i_\mu p^j_\nu+p^i_\nu p^j_\mu)$.

Substitution of Eqs. (3.18)–(3.20) into (3.10) yields the effective potential

$$V^{(1)}_{\rm eff}=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^dk}{(2\pi)^d}\ln\det(\mathscr{D}^{(n)-1}_{ij})=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^dq}{(2\pi R)^d}[\ln(q^2+n^2)^{(d+1)(d-2)/2}+\ln(n^2)^{(d+1)}]+\text{const}+\text{ghost contribution}\ . \tag{3.21}$$

The ghost determinant in (3.10) generates terms similar to the last term in the square brackets in (3.21), which vanishes upon integration (see below). Thus ghosts do not contribute to $V^{(1)}_{\rm eff}$ in the cylindrical gauge.[6]

In the above, we have defined the dimensionless momentum $q=Rk$. Note that the expression in Eq. (3.21) clearly exhibits the presence of $(d+1)(d-2)/2$ propagating modes and $d+1$ nonpropagating (or constrained) modes in each sector. The final integral is, of course, divergent. We will use dimensional regularization[20] (actually its generalization, analytic regularization[21]) to handle the divergence. In particular, it can be shown that[21]

$$\int d^dq=\int d^dq\,f(q)=0 \tag{3.22}$$

for any polynomial $f(q)$, so that Eq. (3.21) simplifies to

$$V^{(1)}_{\rm eff}=\frac{C_d}{(2\pi R)^d}\ , \tag{3.23a}$$

where

$$C_d=\frac{(d+1)(d-2)}{2}\sum_{n=1}^{\infty}\int d^dq\ln(q^2+n^2)\ , \tag{3.23b}$$

which coincides precisely with the calculation of Appelquist and Chodos[6] for $d=4$. In the Appendix it is shown that

$$C_d=-\frac{(d+1)(d-2)}{2(4\pi)^{d/2}}\frac{\Gamma(d+1)}{\Gamma(d/2+1)}\zeta(d+1)\ , \tag{3.24}$$

where $\zeta(z)$ is the Riemann ζ function[22] which is greater than zero for $z>1$, z real, from which we conclude that

$$C_d<0\ \text{ for }\ d=3,4,\ldots\ . \tag{3.25}$$

Thus $V^{(1)}_{\rm eff}(R)$ is attractive for any normal space of dimension greater than 2.

The result for odd dimension d is interesting in two ways. It has been argued[21] that in the early Universe, when the temperature is extremely high, the effective number of space-time dimensions is reduced by one, in which case the physically relevant result would be $d=3$ instead of $d=4$.

The other reason the odd-dimensional result is interesting concerns the relation between the ultraviolet (UV) finiteness of the theory and the total space-time dimension D. In particular, when the method of dimensional regularization is used, a quantum field theory is UV finite if D is odd, but is infinite if D is even. In the present case $D=d+1$, so the theory should be UV finite when d is even and infinite when d is odd. However, the result for C_d is such that it is finite whether d is even or odd. This is easily understood. Despite the nontrivial topology of the compact dimension, the $M^d\times S^1$ background is flat. For a generic field theory in D dimensions, evaluation of the one-loop effective action will yield divergences proportional to the $(D/2)$th power of a curvature invariant.[23] For the present theory in $d+1$ dimensions, the only such invariants available are those built from the Riemann tensor, and hence vanish whether d is even or odd. In this paper one works instead with the tower of reduced d-dimensional theories, but consistency with the $(d+1)$-dimensional approach demands the same result. Indeed, Toms[23] has shown that for consistency it is necessary in dimensional regularization to perform the mode sum before expanding in ϵ, as is done in the Appendix.

IV. A ONE-PARAMETER FAMILY OF GAUGES

We now consider a set of gauges in which the graviton and scalars are coupled:

$$\partial_y(\sqrt{-g}\,)=0\ , \tag{4.1}$$

$$\partial_y(A_\mu)=0\ . \tag{4.2}$$

Because $g_{\mu\nu}$ is a function of the Weyl factor [see Eq. (2.1)], Eq. (4.1) actually represents a family of gauges characterized by the parameter S. This becomes apparent when one calculates the Faddeev-Popov determinant:

$$\det \Delta_{\mathrm{FP}}=\det\left|\frac{\delta F^A(h,\psi,A_\mu)}{\delta\Sigma^B}\right|\ , \tag{4.3}$$

where

$$F^\mu\equiv\frac{\partial A_\mu}{\partial y}=\frac{\partial}{\partial y}(\bar{h}_{\mu D}/\phi_0) \tag{4.4a}$$

and

$$F^D=\frac{\partial}{\partial y}\sqrt{-g}\sim\frac{1}{2}\frac{\partial}{\partial y}h=\frac{1}{2}\frac{\partial}{\partial y}\left[\bar{h}-\frac{Sd}{S+1}\bar{h}_{DD}\right] \tag{4.4b}$$

and Σ^B are the gauge parameters given in Eq. (3.3). Using Eqs. (3.4) and (3.6) we find that

$$\ln\det\left|\frac{\delta F^A}{\delta\Sigma^B}\right|=\sum_{n=-\infty}^{\infty}\int\frac{d^dk}{(2\pi)^d}\ln\det\begin{bmatrix} {m_n}^2 & 0 & 0 & 0 & \cdots & 2ik_0m_n\\ 0 & {m_n}^2 & 0 & 0 & \cdots & 2ik_1m_n\\ 0 & 0 & {m_n}^2 & 0 & \cdots & 2ik_2m_n\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ -ik_0m_n & -ik_1m_n & -ik_2m_n & -ik_3m_n & \cdots & \dfrac{2Sd{m_n}^2}{S+1}\end{bmatrix}$$

$$=\sum_{n=-\infty}^{\infty}\int\frac{d^dk}{(2\pi)^d}\ln\left[{m_n}^{2d}\left[k^2+\frac{Sd}{S+1}{m_n}^2\right]\right]\ . \tag{4.5}$$

We defer evaluation of this integral until the end, but note that after regularization this will, in general, give a nonzero contribution which depends on both S and d.

We now evaluate $\frac{1}{2}\ln\det\overline{\mathscr{D}}^{-1}{}_{ij}$. As in the previous section, we implement the $(\partial_yA^\mu)=0$ gauge condition via a δ function in the functional integral, and integrate over the functions $A_\mu^{(n)}(x)$, so that the net effect is to set $A_\mu^{(n)}=0$, $n\neq 0$ in the effective Lagrangian. The remaining gauge covariance will be handled by adding the gauge-fixing term:

$$\begin{aligned} I_{\mathrm{gf}}&=\frac{1}{2\alpha}\int d^dx\,dy(\partial_y\sqrt{-g}\,)^2\\ &=\frac{1}{2\alpha}\int d^dx\sum_{n=-\infty}^{\infty}{m_n}^2[h^{(n)}(x)]^2+O(\epsilon^3)\ . \end{aligned} \tag{4.6}$$

Since we have not explicitly eliminated $\psi^{(n)}$ from the functional integral, the inverse propagator in each sector has the following block structure:

$$\overline{\mathscr{D}}{}^{(n)-1}_{ij}=\begin{bmatrix}\overline{\mathscr{D}}{}^{(n)-1}_{\mu\nu,\lambda\sigma} & (\mathscr{D}^{(n)-1}_{\psi h})_{\lambda\sigma}\\ (\mathscr{D}^{(n)-1}_{h\psi})_{\mu\nu} & \mathscr{D}^{(n)-1}_{\psi\psi}\end{bmatrix}\ , \tag{4.7}$$

where

$$(\mathscr{D}^{(n)-1}_{h\psi})_{\mu\nu}=(\mathscr{D}^{(n)-1}_{\psi h})_{\mu\nu}=-a_3k_\mu k_\nu+(a_3k^2+a_4{m_n}^2)\delta_{\mu\nu}\ , \tag{4.8}$$

$$\overline{\mathscr{D}}{}^{(n)-1}_{\mu\nu,\lambda\sigma}=\mathscr{D}^{(n)-1}_{\mu\nu,\lambda\sigma}+\frac{{m_n}^2}{\alpha}\delta_{\mu\nu}\delta_{\lambda\sigma} \tag{4.9}$$

with $\mathscr{D}^{(n)-1}_{\mu\nu,\lambda\sigma}$ given by Eq. (3.14). Thus, in this gauge the graviton and scalar are coupled.

The appropriate basis for calculating the determinant is

$$X^{(n)}=\begin{bmatrix} f^{(n)} \\ X^{(n)}_{\mu\nu} \end{bmatrix}, \tag{4.10}$$

where $X^{(n)}_{\mu\nu}$ is expanded as in Eq. (3.15) and f is a scalar function. In this basis

$$\overline{\mathscr{D}}^{(n)-1}=\begin{bmatrix} \overline{\mathscr{D}}^{(n)-1}_1 & \\ & \overline{\mathscr{D}}^{(n)-1}_2 \end{bmatrix}, \tag{4.11}$$

where $\overline{\mathscr{D}}^{(n)-1}_2$ is again given in Eq. (3.20) but now

$$\overline{\mathscr{D}}^{(n)-1}_1=\begin{bmatrix} 2(-a_1k^2+a_2m_n{}^2) & a_4m_n{}^2 & (d-1)(a_3k^2+a_4m_n{}^2) \\ a_4m_n{}^2 & m_n{}^2/\alpha & (d-1)(\frac{1}{2}m_n{}^2+m_n{}^2/\alpha) \\ a_3k^2+a_4m_n{}^2 & \frac{1}{2}m_n{}^2+m_n{}^2/\alpha & (d-1)m_n{}^2/\alpha+\dfrac{(d-2)}{2}(m_n{}^2+k^2) \end{bmatrix}. \tag{4.12}$$

The one-loop effective potential is thus

$$\begin{aligned} V^{(1)}_{\rm eff} &= \tfrac{1}{2}\ln\det\overline{\mathscr{D}}^{\,-1}-\ln\det\Delta_{\rm FP} \\ &=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^dq}{(2\pi R)^d}\ln\left[\frac{d-1}{4\alpha}(S+1)^2\left[q^2+\frac{Sd}{S+1}n^2\right]^2(q^2+n^2)^{(d+1)(d-2)/2}(n^2)^{(d+1)}\right] \\ &\quad -\sum_{n=-\infty}^{\infty}\int\frac{d^dq}{(2\pi R)^d}\ln\left[n^{2d}\left[q^2+\frac{Sd}{S+1}n^2\right]\right]. \end{aligned} \tag{4.13}$$

Note that the first determinant, $\det\overline{\mathscr{D}}^{\,-1}$, vanishes as expected in the limit $\alpha\to\infty$ reflecting the singularity in $\overline{\mathscr{D}}$ when the gauge-fixing term is removed. Although both this term and the Faddeev-Popov term depend nontrivially on S and d, we see that all gauge-dependent contributions do either cancel or drop out so that the total one-loop effective potential after regularization is exactly the same as in Eq. (3.21).

V. CONCLUSIONS

We have calculated the one-loop effective potential in $D=(d+1)$-dimensional Kaluza-Klein theory compactified to $M^d\times S^1$ in a family of gauges characterized by the Weyl exponent S. In addition to verifying explicitly the invariance of the one-loop effective potential under infinitesimal gauge changes (i.e., changes in S), our calculation also clarifies the role played by the Weyl factor in the quantum dynamics of Kaluza-Klein theory, and confirms an earlier claim[6] that it is a useful bookkeeping device but does not affect physical quantities.

Cohler and Chodos[10] have proven a theorem stating that the one-loop effective potential is invariant under infinitesimal gauge charges when it is evaluated at a solution of the classical field equations. Since our gauges fall into the class of gauges considered by Cohler and Chodos, their theorem requires the effective potential we have calculated to be independent of S. Moreover, because Eq. (4.4b) has the limit

$$\lim_{s\to-1}F^D=\left[\lim_{s\to-1}F^D-\frac{1}{2}\frac{Sd}{S+1}\right]\frac{\partial}{\partial y}\psi, \tag{5.1}$$

the cylindrical gauge is actually a special case of our family of gauges. The theorem of Cohler and Chodos also requires that our result agrees with that of Applequist and Chodos,[6] as indeed it does.

Although the Casimir effect in Kaluza-Klein theory points out semiclassical instabilities of classical compactifications, it is but the first step of a more interesting type of calculation[5,8,12] which allows the prediction of coupling constants and stable radii by finding self-consistent solutions to the one-loop corrected equations of motion. Because in such calculations the one-loop effective potential must be evaluated off-shell, it is not clear that any of the theorems concerning the gauge invariance of the effective potential apply. In fact, recent calculations by Randjbar-Daemi and Sarmadi[12] in the light-cone gauge seem to imply that gauge invariance may not hold. The gauges discussed in the present paper provide a useful tool for analyzing the gauge dependence of the self-consistent one-loop calculations. The results of this analysis will be presented elsewhere.

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APPENDIX

We compute the quantity

$$A=\sum_{n=1}^{\infty}\int d^d q\,\ln(q^2+n^2) \tag{A1}$$

which, on account of its divergence, calls for a regularization. We use dimensional regularization.

Consider the known integral

$$\begin{aligned} I(\omega,\mu,a)&=\int d^{2\omega}q\,(q^2+a)^{\mu},\quad \mathrm{Re}(a)>0\\ &=\pi^{\omega}a^{\omega+\mu}\Gamma(-\mu-\omega)/\Gamma(-\mu)\ . \end{aligned} \tag{A2}$$

Then A may be defined as

$$A=\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\lim_{\mu\to 0}\frac{\partial}{\partial\mu}I(d/2+\epsilon,\mu,n^2)\ . \tag{A3}$$

Because

$$\lim_{\mu\to 0}1/\Gamma(-\mu)=0$$

but

$$\lim_{\mu\to 0}\frac{\partial}{\partial\mu}[1/\Gamma(-\mu)]=-1\ , \tag{A4}$$

the factor $\Gamma(-\mu)$ in (A2) is the only one that needs to be differentiated, and

$$A=-\lim_{\epsilon\to 0}\sum_{n=1}^{\infty}\pi^{\omega}n^{2\omega}\Gamma(-\omega)\ , \tag{A5}$$

where $\omega=d/2+\epsilon$. The summation over n yields the Riemann ζ function, formally defined as

$$\zeta(z)=\sum_{n=1}^{\infty}n^{-z} \tag{A6}$$

and extended to $\mathrm{Re}(z)<1$ by analytic continuation.

Thus

$$\begin{aligned} A&=-\lim_{\epsilon\to 0}\pi^{d/2+\epsilon}\zeta(-d-2\epsilon)\Gamma(-d/2-\epsilon)\\ &=\begin{cases} -\pi^{d/2}\zeta(-d)\Gamma(-d/2), & d\ \text{odd}\\ -\dfrac{(-)^{d/2}\pi^{d/2}}{(d/2)!}\lim\limits_{\epsilon\to 0}\left[(1+\epsilon\ln\pi)2\epsilon\zeta'(-d)\left[\dfrac{1}{\epsilon}-\psi(d/2+1)\right]+O(\epsilon)\right]=-\dfrac{(-)^{d/2}\pi^{d/2}}{(d/2)!}2\zeta'(-d), & d\ \text{even} \end{cases} \end{aligned} \tag{A7}$$

where ψ is the digamma function, and the property that $\zeta(-2m)=0$ for $m=1,2,\ldots$ has been used.

Using the identity

$$\zeta(z)=2^z\pi^{z-1}\sin\left[\frac{\pi z}{2}\right]\Gamma(1-z)\zeta(1-z) \tag{A8}$$

we obtain for all integral d

$$A=-\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d+1)}{\Gamma(d/2+1)}\zeta(d+1)\ . \tag{A9}$$

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