

Two-dimensional Coulomb gas studied in the sine-Gordon formulation

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(Received 18 September 1989)

Based on the sine-Gordon field-theory formulation, the two-dimensional Coulomb-gas system is studied using a Gaussian wave-functional or a Gaussian effective-potential method. In the point-charge case the results are presented analytically, and the expressions for the critical line are derived.

I. INTRODUCTION

Since the pioneering work of Kosterlitz-Thouless (K-T) on phase transitions,¹ investigations of the two-dimensional Coulomb gas (2D CG) have attracted much attention. (For an excellent review, see Ref. 2.) Among the various approaches to this problem, the sine-Gordon (SG) field theory occupies an important position.³⁻⁷ In 1985, Minnhagen⁸ combined the SG formulation with a very physical picture of self-consistent linear screening to obtain an improved Kosterlitz renormalization-group equation.⁹

In this paper, we report the results of explicit calculations obtained by applying to this problem variational, and therefore nonperturbative, methods used in relativistic SG field theory.¹⁰⁻¹⁴ The paper is organized as follows. In Sec. II, the SG formulation will be realized in a Gaussian wave-functional method. In Sec. III, an essentially equivalent but more flexible method, the Gaussian effective-potential method, will be presented. In Secs. IV and V we discuss the physical implications of the aforementioned calculations, including the critical line, the phase diagram, and the order parameters. Section VI contains the summary and discussion. Some mathematical details are given in the Appendixes.

II. VARIATION WITH A GAUSSIAN WAVE FUNCTIONAL

For describing the 2D CG, one starts from a nonlocal SG theory with the Hamiltonian density⁸

$$\mathcal{H}_{\text{SG}} = \frac{1}{2}(\nabla\phi)^2 - \frac{2z}{\xi^2} \cos\beta\bar{\phi}(\mathbf{r}), \quad (2.1)$$

where $\phi(\mathbf{r}) = \phi(x, y)$ is a real scalar field,

$$\bar{\phi}(\mathbf{r}) = \int d^2r' f_{\xi}(|\mathbf{r} - \mathbf{r}'|) \phi(\mathbf{r}'); \quad (2.2)$$

$f_{\xi}(\mathbf{r})$ describes the normalized spatial charge distribution of particles in the 2D CG, ξ is a length scale that we later use for the ultraviolet cutoff [see (2.33)], $z = \exp(-E_c/T)$ is the fugacity, with E_c being the nonelectrostatic part of the self-energy of each particle that is assumed to be the same for both positive and negative charges in the neutral 2D CG. The temperature T (with $k_B=1$) will be expressed in units of charge squared, which in 2D has the dimension of energy. Thus z , $\beta = \sqrt{2\pi}/T$, and ϕ are all dimensionless.

In the SG theory, the expectation value of a functional of ϕ , $\mathcal{O}[\phi]$, is given by the path integral^{2,8}

$$\langle \mathcal{O}[\phi] \rangle_{\text{SG}} = \frac{\int \mathcal{D}\phi \exp \left[- \int d^2r \mathcal{H}_{\text{SG}} \right] \mathcal{O}[\phi]}{\int \mathcal{D}\phi \exp \left[- \int d^2r \mathcal{H}_{\text{SG}} \right]}. \quad (2.3)$$

The absence of a kinetic term in (2.1) allows one to view $\mathcal{H}_{\text{SG}}$ as a Lagrangian density $\mathcal{L}_E$ in 2D Euclidean space if one identifies the coordinate y with vt_E , where t_E is the Euclidean time and v is the typical speed, whose value will be shown to be irrelevant in later calculations. Then (2.3) could be looked as a "quantum average" in a Euclidean field theory, where the Planck constant $\hbar$ is

substituted with $1/v$. We now take $\mathcal{L}_E$ and go to Minkowski (1+1) space, then the Lagrangian density reads

$$\mathcal{L} = -\frac{1}{2} \left[\frac{\partial \phi}{\partial x} \right]^2 + \frac{1}{2v^2} \left[\frac{\partial \phi}{\partial t} \right]^2 + \frac{2z}{\xi^2} \cos \beta \tilde{\phi} \quad (2.4)$$

with

$$\tilde{\phi}(x, t) = \int dx' f_{\xi}(x - x') \phi(x', t), \quad (2.5)$$

where $f_{\xi}(x - x')$ is the one-dimensional charge distribution function that could be viewed as the projection of the two-dimensional one defined in (2.2).

After defining the canonical momentum conjugate to ϕ ,

$$\pi_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{v^2} \dot{\phi}, \quad (2.6)$$

one obtains the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} v^2 \pi_{\phi}^2 + \frac{1}{2} \left[\frac{\partial \phi}{\partial x} \right]^2 - \frac{2z}{\xi^2} \cos \beta \tilde{\phi}. \quad (2.7)$$

One can now quantize¹⁵ the scalar field by imposing the canonical commutation relation

$$[\hat{\phi}(x, t), \hat{\pi}_{\phi}(x, t)] = i v^{-1} \delta(x - x'), \quad (2.8)$$

which in turn is realized via the operator form

$$\hat{\pi}_{\phi} = -i v^{-1} \frac{\delta}{\delta \phi}. \quad (2.9)$$

With the Hamiltonian (2.7) thus quantized replacing $\mathcal{H}_{SG}$, the expectation value of $\hat{\mathcal{O}}[\hat{\phi}, \hat{\pi}_{\phi}]$ is then replaced by

$$\langle \hat{\mathcal{O}}[\hat{\phi}, \hat{\pi}_{\phi}] \rangle_{SG} \rightarrow \langle \Psi | \hat{\mathcal{O}}[\hat{\phi}, \hat{\pi}_{\phi}] | \Psi \rangle, \quad (2.10)$$

where Ψ is the solution of the functional eigenequation

$$\hat{H} \Psi = E \Psi. \quad (2.11)$$

The exact form of Ψ is of course difficult to obtain, here we confine ourselves to the ansatz of restricting the wave functional Ψ to be of Gaussian type:^{10,13,14}

$$\Psi = N_g \exp \left[i v \int dx \mathcal{P}_x \phi(x) - \frac{v}{2} \int dx dy (\phi_x - \Phi_x) g(x - y) (\phi_y - \Phi_y) \right], \quad (2.12)$$

$$\langle \Psi | \Psi \rangle \equiv \int \mathcal{D}\phi \Psi^* \Psi = 1, \quad \Phi(x) \equiv \Phi_x = \langle \Psi | \phi_x | \Psi \rangle, \quad (2.12a)$$

where the correlation functions $g(x - y)$ and $\mathcal{P}_x = \langle \Psi | \pi_{\phi_x} | \Psi \rangle$ are treated as variables to be determined by a variational procedure that minimizes the energy E ,

$$E = \langle \Psi | \hat{H} | \Psi \rangle = \int dx \langle \Psi | \hat{\mathcal{H}} | \Psi \rangle. \quad (2.13)$$

Using techniques outlined in Appendix A, one obtains

$$E = \int dx \left\{ \left[\frac{v^2}{2} \mathcal{P}_x^2 + \frac{1}{2} \left[\frac{\partial \Phi}{\partial x} \right]^2 + \frac{v}{4} g_{xx} \right] - \frac{1}{4v} \int dy \delta(x - y) \partial_x^2 g_{xy}^{-1} - \frac{2z}{\xi^2} \cos \beta \tilde{\Phi}_x \exp(-\frac{1}{4} \beta^2 v^{-1} \tilde{g}_{xx}^{-1}) \right\}, \quad (2.14)$$

where

$$\tilde{\Phi}_x \equiv \tilde{\Phi}(x) = \int dx' \Phi(x') f_{\xi}(x - x'),$$

$g_{xy} \equiv g(x - y)$, and $g^{-1}(x - y) \equiv g_{xy}^{-1}$ are defined via the relation

$$\int dy g(x - y) g^{-1}(y - z) = \delta(x - z) \quad (2.15)$$

so that

$$\tilde{g}_{xx}^{-1} = \int \int f_{\xi}(x - y) g_{yz}^{-1} f_{\xi}(z - x') dy dz. \quad (2.16)$$

Define Fourier transformations

$$f_{\xi}(x - y) = \int \frac{dp}{2\pi} f(p) e^{ip(x - y)}, \quad (2.17)$$

$$g(x - y) = \int \frac{dp}{2\pi} g(p) e^{ip(x - y)}, \quad (2.18)$$

we have

$$g^{-1}(x - y) = \int \frac{dp}{2\pi} \frac{1}{g(p)} e^{ip(x - y)}, \quad (2.19)$$

$$\tilde{g}_{xy}^{-1} = \int \frac{dp}{2\pi} \frac{f^2(p)}{g(p)} e^{ip(x - y)}. \quad (2.20)$$

Variation of E with respect to $\mathcal{P}_x$ leads to $\mathcal{P}_x = 0$.

We now treat $g(p)$ as a variable, then the condition

$$\frac{\delta E}{\delta g(k)} = 0 \quad (2.21)$$

gives

$$v^2 g^2(k) = k^2 + \Delta_k^2 \cos \beta \tilde{\Phi}, \quad (2.22)$$

where

$$\Delta_k^2 = \Delta_0^2 f^2(k), \quad \Delta_0^2 = \alpha e^{-I}, \quad I = \frac{\beta^2}{4v} \int \frac{dp}{2\pi} \frac{f^2(p)}{g(p)}, \quad (2.23)$$

where $\alpha \equiv 2z\beta^2\xi^{-2}$. Thus in the space uniform configuration the energy density ϵ , that is the integrand in (2.14), is

$$\begin{aligned}
\varepsilon &= \frac{1}{4} \int \frac{dp}{2\pi} M(p, \tilde{\Phi}) + \frac{1}{4} \int \frac{dp}{2\pi} p^2 / M(p, \tilde{\Phi}) - \frac{2z}{\xi^2} e^{-I} \cos \beta \tilde{\Phi}, \\
&= \frac{1}{2} \int \frac{dp}{2\pi} M(p, \tilde{\Phi}) - \frac{1}{4} \Delta_0^2 \cos \beta \tilde{\Phi} \int \frac{dp}{2\pi} f^2(p) / M(p, \tilde{\Phi}) \\
&\quad - \frac{2z}{\xi^2} e^{-I} \cos \beta \tilde{\Phi}, \\
M(p, \tilde{\Phi}) &\equiv (p^2 + \Delta_0^2 \cos \beta \tilde{\Phi})^{1/2}.
\end{aligned} \tag{2.24}$$

The further variational condition

$$\left. \frac{\partial \varepsilon}{\partial \Delta_0^2} \right|_{\Delta_0^2 = \Omega^2} = 0 \tag{2.25}$$

reproduces

$$\Delta_0^2 = \Omega^2 = \alpha e^{-I} \tag{2.26}$$

in conformity with (2.23). The stability criterion then gives

$$\left. \frac{\partial^2 \varepsilon}{\partial (\Delta_0^2)^2} \right|_{\Delta_0^2 = \Omega^2} = \frac{1}{8} \int \frac{dp}{2\pi} f^4(p) M^{-3}(p, \tilde{\Phi}) \left[1 - \frac{\beta^2}{8} \Delta_0^2 \cos \beta \tilde{\Phi} \int \frac{dp}{2\pi} f^4(p) M^{-3}(p, \tilde{\Phi}) \right] \geq 0 \tag{2.27}$$

For simplicity, we consider the limit for point charge, i.e., $f_\xi(x) \rightarrow \delta(x)$, or $f(p) \rightarrow 1$. Then Eq. (2.27) simplifies to

$$\frac{2\pi}{T} = \beta^2 \leq 8\pi \quad \text{or} \quad T > T_c = \frac{1}{4}. \tag{2.28}$$

The critical value of coupling constant $\beta_c^2 = 8\pi$ in the SG field theory, first discovered by Coleman,^{16,10,14} now implies a critical temperature of phase transition in 2D CG at least for $z \rightarrow 0$ (see further).

Being a function of Φ and $\Delta_0^2(\Phi)$, the energy density should take a minimum value with respect to Φ . To this purpose, also consider the $f(p) = 1$ case, we first notice that

$$\frac{d\Delta_0^2}{d\Phi} = -\Delta_0^2 \frac{dI}{d\Phi} \tag{2.29}$$

then

$$\frac{d\Delta_0^2}{d\Phi} = (\beta^2 \Delta_0^2 J \sin \beta \Phi) / (\beta^2 \Delta_0^2 J \cos \beta \Phi - 8), \tag{2.30}$$

where

$$J = \int \frac{dp}{2\pi} (p^2 + \Delta_0^2 \cos \beta \Phi)^{-3/2} > 0. \tag{2.31}$$

Thus the equation

$$\frac{d\varepsilon}{d\Phi} = (\frac{1}{8} \beta \Delta_0^4 J \cos \beta \Phi + \alpha \beta^{-2} e^{-I}) \sin \beta \Phi$$

indicates that $\Phi = 0$ corresponds to a stable phase as

$$\left. \frac{d^2 \varepsilon}{d\Phi^2} \right|_{\Phi=0} = \alpha e^{-I} + \frac{1}{8} \beta^2 \Delta_0^4 J > 0. \tag{2.32}$$

However, to perform the concrete calculation one has to be very careful. Because $f(p) = 1$ for the point-charge case, an ultraviolet cutoff in momentum integration is needed. Since there is already a small length scale, ξ , in the theory, for simplicity, we use ξ^{-1} as the ultraviolet cutoff. This implies that we are not interested in what happens at a length scale less than ξ . Similarly, as we will see, because $\Delta_0^2 \rightarrow 0$ when $T \rightarrow T_c + 0^+$, an infrared

cutoff λ_c^{-1} is also needed. Hence, for example, the integral in (2.23) reads

$$\begin{aligned}
I &= \frac{\beta^2}{4\pi} \int_{\lambda_c^{-1}}^{\xi^{-1}} dp (p^2 + \Delta_0^2)^{-1/2} \\
&= \frac{\beta^2}{4\pi} \ln \frac{\xi^{-1} + (\Delta_0^2 + \xi^{-2})^{1/2}}{\lambda_c^{-1} + (\lambda_c^{-2} + \Delta_0^2)^{1/2}}.
\end{aligned} \tag{2.33}$$

Instability sets in when the right-hand side of (2.32) decreases to zero. To find the condition for instability we send $\lambda_c^{-1} \rightarrow 0$ and reexpress (2.33) as a function of $\eta \equiv \Delta_0^2 \xi^2$:

$$\eta - \left[\frac{4\pi z}{T} \right]^{4T/(4T-1)} [2 + 2(1+\eta)^{1/2} + \eta]^{-1/(4T-1)} = 0. \tag{2.34}$$

For some fixed values of E_c in $z = e^{-E_c/T}$ and varying the temperature T , we have calculated numerically the roots of (2.34). The general trend is as follows. For sufficiently small z , and for temperature interval $(\frac{1}{4}, T_u)$, there are roots in the interval $0 < \eta < 1$. As z increases, or equivalently, as E_c decreases for a given T , the upper bound T_u of the temperature interval decreases. When z increases to $\approx 3/8\pi$, (2.34) has no root in the interval $0 < \eta < 1$. The value of η approaches to zero smoothly when $T \rightarrow \frac{1}{4} + 0^+$. This implies the system undergoes a phase transition at $T = T_c \approx \frac{1}{4}$. When $\eta \ll 1$, Eq. (2.34) is simplified to

$$\eta = 4 \left[\frac{\pi z}{T} \right]^{4T/(4T-1)} \tag{2.34a}$$

Above the phase transition, the energy density of the SG system, ε , can be evaluated from (2.24), with $f(p) = 1$, to be

$$\varepsilon \xi^2 = \frac{1}{4\pi} (1 + \eta \cos \beta \Phi)^{1/2} - \eta \beta^{-2} \cos \beta \Phi. \tag{2.35}$$

If $\Delta_0^2 \xi^2 \ll 1$, one has, approximately,

$$\varepsilon \xi^2 \approx \frac{1}{4\pi} + \left[\frac{1}{8\pi} - \beta^{-2} \right] \eta \cos \beta \Phi. \quad (2.36)$$

Thus we see once again that under the critical value of β^2 , $\beta_c^2 = 8\pi$, $\Phi = 0$ corresponds to the stable phase with lowest energy.

We are now in a position to evaluate the linearly screened potential $V_L(r)$ related to the one-particle Green's function $G(r)$ in the SG theory:^{8,2}

$$V_L(r) = 2\pi G(r) = 2\pi \langle \phi(r) \phi(0) \rangle_{\text{SG}}. \quad (2.37)$$

Recall that in this method the spatial dimensions have been lowered from 2 to 1. However, assuming the 2D area is large enough and the system is isotropic, we may replace the argument r by x or vice versa. Thus after using the methods described in Appendix A, one finds, with $f(p) = 1$ and $\Phi = 0$,

$$G(r) = \frac{1}{2} v^{-1} g_{r0}^{-1} = \frac{1}{2} \int \frac{dp}{2\pi} \frac{e^{ipr}}{(p^2 + \Delta_0^2)^{1/2}} = \frac{1}{2\pi} K_0(\Delta_0 r), \quad (2.38)$$

where $K_0(\Delta_0 r)$ is the MacDonald function with the

$$\langle \Delta n(r) \Delta n(0) \rangle = 2z^2 \xi^{-4} \exp \left[-\frac{\beta^2}{2} v^{-1} \bar{g}_{00}^{-1} \right] (e^{-(\beta^2/2)v^{-1} \bar{g}_{r0}^{-1}} - e^{(\beta^2/2)v^{-1} \bar{g}_{r0}^{-1}}) \quad (r \neq 0). \quad (2.42)$$

Comparing to (2.40), one sees that

$$z_{\text{eff}} = z \exp \left[-\frac{\beta^2}{4} v^{-1} \bar{g}_{00}^{-1} \right], \quad (2.43)$$

$$U_{\text{eff}}^{++}(r) = U_{\text{eff}}^{--}(r) = \pi v^{-1} \bar{g}_{r0}^{-1}.$$

In the point-charge limit, with $f(p) = 1$, one finds that²

$$n_F = \frac{2z_{\text{eff}}}{\xi^2} = \frac{2z}{\xi^2} e^{-I} = \frac{\Delta_0^2}{\beta^2} = \frac{T \Delta_0^2}{2\pi} \quad (2.44)$$

is just the free charge density [see (5.5) in Sec. V], while⁸

$$\langle \Delta n(r) \Delta n(\xi) \rangle \rightarrow -\frac{4z^2}{\xi^4} \left[\frac{\Delta_0 \xi}{2} \right]^{1/T} K_0(\Delta_0 r) \sim -\frac{2z^2}{\xi^4} \left[\frac{\pi z}{T} \right]^{(2-T)/(4T-1)} \left[\frac{\pi \xi}{r} \right]^{1/2} \exp \left[-2 \left[\frac{\pi z}{T} \right]^{2T/(4T-1)} \left[\frac{r}{\xi} \right] \right]. \quad (2.46)$$

In the opposite extreme case, $\Delta_0 r \ll 1$,

$$\langle \Delta n(r) \Delta n(\xi) \rangle \longrightarrow -\frac{4z^2}{\xi^4} \left[\frac{\xi}{r} \right]^{1/T}, \quad (T > T_c). \quad (2.47)$$

Moreover, in the vicinity of phase transition ($T \rightarrow \frac{1}{4} + 0^+$),

asymptotic behavior

$$V_L(r) \approx \left[\frac{\pi}{2\Delta_0 r} \right]^{1/2} e^{-\Delta_0 r} = \left[\frac{\pi \lambda}{2r} \right]^{1/2} e^{-r/\lambda} \quad (r \gg \lambda). \quad (2.39)$$

Now it is clear that $\lambda \equiv 1/\Delta_0$ is just the Debye screening length in the high-temperature plasma (P) phase. A finite λ implies a short-range order in this phase. When $\lambda \rightarrow \lambda_c \rightarrow \infty$ as $T \rightarrow T_c$ from above, the system undergoes a charge-binding transition and becomes a dipole (D) phase.

Finally, the point-charge density correlation function, expressed as²

$$\langle \Delta n(r) \Delta n(0) \rangle = 2z_{\text{eff}}^2 \xi^{-4} (e^{-U_{\text{eff}}^{++}(r)/T} - e^{U_{\text{eff}}^{--}(r)/T}) \quad (2.40)$$

and can be calculated in SG formulation to be

$$\langle \Delta n(r) \Delta n(0) \rangle = -4z^2 \xi^{-4} \langle \sin[\beta \tilde{\phi}(r)] \sin[\beta \tilde{\phi}(0)] \rangle_{\text{SG}} + 2\delta(r) \xi^{-2} z \langle \cos[\beta \tilde{\phi}(0)] \rangle_{\text{SG}} \quad (2.41)$$

Using Eq. (A9), we have

$$\langle \Delta n(r) \Delta n(0) \rangle = -\frac{2z^2}{\xi^4} \exp \left[\frac{1}{T} [V_L(r) - V_L(0)] \right] \times \left[1 - \exp \left[-\frac{2}{T} V_L(r) \right] \right] \quad (r \neq 0), \quad (2.45)$$

with $V_L(r) = K_0(\Delta_0 r)$. When $\Delta_0 r \gg 1$, $T > T_c = \frac{1}{4}$, and using (2.34a), one has

$$\langle \Delta n(r) \Delta n(\xi) \rangle \xrightarrow{T \rightarrow \frac{1}{4}, r \ll \Delta_0^{-1}} -\frac{4z^2}{r^4}, \quad (2.48)$$

which exhibits a quasi-long-range of order of charge density correlation. Further discussion will be postponed until Sec. V.

III. METHOD OF GAUSSIAN EFFECTIVE POTENTIAL

Whereas the method described in the previous section has the advantage of being more intuitive, it suffers from the necessity of sacrificing one spatial dimension and does

not allow simple improvements in calculational accuracy. In this section we wish to propose an essentially equivalent but more flexible method that does not require a lowering of spatial dimensions.

Our tactic is as follows.¹² Beginning from the generating functional of the SG system

$$Z_J^{\text{SG}} = e^{-W[J]} = \int \mathcal{D}\phi e^{-S_J} \\ = \int \mathcal{D}\phi \exp \left[- \int d^2r \left[\frac{1}{2} \phi (-\nabla^2 + M^2) \phi - \frac{1}{2} M^2 \phi^2 - \frac{2z}{\xi^2} \cos \beta \tilde{\phi} \right] + \int d^2r J(\mathbf{r}) \tilde{\phi}(\mathbf{r}) \right], \quad (3.1)$$

and defining

$$\Phi(\mathbf{r}) = \frac{1}{Z_J^{\text{SG}}} \int \mathcal{D}\phi \phi(\mathbf{r}) e^{-S_J} \\ = \frac{\delta}{\delta \tilde{J}(\mathbf{r})} \ln Z_J^{\text{SG}} = - \frac{\delta W[J]}{\delta \tilde{J}(\mathbf{r})}, \quad (3.2)$$

and assuming Φ is uniform in space, one can find the effective potential $V_{\text{eff}}(\Phi)$ via a Legendre transformation:

$$\Gamma[\Phi] \equiv \int d^2r V_{\text{eff}}(\Phi) = W[J] + \int d^2r \tilde{J}(\mathbf{r}) \Phi(\mathbf{r}). \quad (3.3)$$

The inserted mass square M^2 is necessary for calculational purposes and will be fixed by a variational procedure (see further). Denoting

$$\phi'(\mathbf{r}) = \phi(\mathbf{r}) - \Phi(\mathbf{r}) \quad (3.4)$$

one may split the action S_J into

$$S_J = S_J^{(0)} + S_J^{(1)} + \epsilon S^{(2)}, \quad (3.5)$$

where

$$S_J^{(0)} = \int d^2r \left[\frac{1}{2} \Phi (-\nabla^2) \Phi - J(\mathbf{r}) \tilde{\Phi}(\mathbf{r}) \right], \quad (3.6)$$

$$S_J^{(1)} = \int d^2r \left[\frac{1}{2} \phi' (-\nabla^2 + M^2) \phi' - J(\mathbf{r}) \tilde{\phi}'(\mathbf{r}) \right], \quad (3.7)$$

$$\epsilon S^{(2)} = \epsilon \int d^2r \left[-\frac{1}{2} M^2 \phi'^2 - \frac{2z}{\xi^2} (\cos \beta \tilde{\Phi} \cos \beta \tilde{\phi}' - \sin \beta \tilde{\Phi} \sin \beta \tilde{\phi}') \right]. \quad (3.8)$$

The dimensionless parameter ϵ , not to be confused with the quantity in (2.24), is introduced for discriminating the order of approximation; it will be set equal to 1 at the final stage of computation. Rewrite ϕ' as ϕ again and denoting an unnormalized average of $\mathcal{O}[\phi]$ as

$$\langle \mathcal{O}[\phi] \rangle_0 = \int \mathcal{D}\phi \mathcal{O}[\phi] \exp(-S_J^{(1)}) \quad (3.9)$$

one can recast (3.1) into

$$Z_J^{\text{SG}} = \exp(-S_J^{(0)}) \langle e^{-\epsilon S^{(2)}[\phi]} \rangle_0. \quad (3.10)$$

The generating functional related to $S_J^{(1)}$ is easily calculated in a manner similar to that described in Appendix A:

$$Z_J^{M^2} \equiv \int \mathcal{D}\phi e^{-S_J^{(1)}} = [\det(-\nabla^2 + M^2)]^{-1/2} \exp \left[\frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2 \tilde{J}(\mathbf{r}_1) h^{-1}(\mathbf{r}_1, \mathbf{r}_2) \tilde{J}(\mathbf{r}_2) \right], \quad (3.11)$$

where

$$h(\mathbf{r}_1, \mathbf{r}_2) = (-\nabla_1^2 + M^2) \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad h^{-1}(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \frac{1}{-\nabla_2^2 + M^2} = \frac{1}{\nabla_1^2 + M^2} \delta(\mathbf{r}_1 - \mathbf{r}_2), \\ \int h^{-1}(\mathbf{r}_1, \mathbf{r}_2) h(\mathbf{r}_2, \mathbf{r}_3) d\mathbf{r}_2 = \delta(\mathbf{r}_1 - \mathbf{r}_3). \quad (3.12)$$

If we confine our calculation to $\mathcal{O}(\epsilon)$, then

$$Z_J^{\text{SG}} \simeq \exp(-S_J^{(0)}) \int \mathcal{D}\phi (1 - \epsilon S^{(2)}[\phi]) e^{-S_J^{(1)}}. \quad (3.13)$$

Substituting (3.8) into (3.13), one needs the following averages:

$$\langle \phi^2(\mathbf{r}) \rangle_0 = \frac{\delta}{\delta \tilde{J}(\mathbf{r})} \frac{\delta}{\delta \tilde{J}(\mathbf{r})} Z_J^{M^2} = Z_J^{M^2} \left[h^{-1}(\mathbf{r}, \mathbf{r}) + \int d\mathbf{r}_1 d\mathbf{r}_2 J(\mathbf{r}_1) J(\mathbf{r}_2) \tilde{h}^{-1}(\mathbf{r}_1, \mathbf{r}) \tilde{h}^{-1}(\mathbf{r}, \mathbf{r}_2) \right], \quad (3.14)$$

$$\langle \cos \beta \tilde{\phi}(\mathbf{r}) \rangle_0 = \text{Re} \langle e^{i\beta \tilde{\phi}(\mathbf{r})} \rangle_0 = \text{Re} [Z_J^{M^2}]_{J(\mathbf{r}') \rightarrow J(\mathbf{r}') + i\beta \delta(\mathbf{r} - \mathbf{r}')} \\ = Z_J^{M^2} e^{-\frac{1}{2} \beta^2 \tilde{h}^{-1}(\mathbf{r}, \mathbf{r})} \cos \left[\beta \int d\mathbf{r}_1 \tilde{h}^{-1}(\mathbf{r}, \mathbf{r}_1) J(\mathbf{r}_1) \right], \quad (3.15)$$

$$\langle \sin \beta \tilde{\phi}(\mathbf{r}) \rangle_0 = \text{Im} \langle e^{i\beta \tilde{\phi}(\mathbf{r})} \rangle_0 = Z_J^{M^2} e^{-\frac{1}{2}\beta^2 \tilde{h}^{-1}(\mathbf{r}, \mathbf{r})} \sin \left[\beta \int d\mathbf{r}_1 \tilde{h}^{-1}(\mathbf{r}_1, \mathbf{r}) J(\mathbf{r}_1) \right]. \quad (3.16)$$

Denoting the determinant in $Z_J^{M^2}$ as

$$\begin{aligned} [\det(-\nabla^2 + M^2)]^{-1/2} &= \exp\left[-\frac{1}{2}\text{Tr} \ln(-\nabla^2 + M^2)\right] \\ &= \exp\left[-\frac{1}{2} \int d^2r D\right] \end{aligned} \quad (3.17)$$

with

$$D = \int \frac{d^2p}{(2\pi)^2} \ln(p^2 + M^2),$$

we can get the effective potential $V_{\text{eff}}(\Phi)$ along the line sketched in (3.1)–(3.3) as

$$V_{\text{eff}}(\Phi, M^2) = \frac{1}{2}D - \frac{1}{2}M^2 h^{-1}(\mathbf{r}, \mathbf{r}) - \frac{2z}{\xi^2} e^{-K \cos \beta \Phi}, \quad (3.18)$$

where

$$h^{-1}(\mathbf{r}, \mathbf{r}) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2 + M^2)} = \frac{\partial}{\partial M^2} D \quad (3.19)$$

[see (3.12)] and

$$\begin{aligned} G(\mathbf{r}) &\equiv \langle \phi(\mathbf{r}) \phi(0) \rangle_{\text{SG}} = \left[\frac{1}{z_J^{\text{SG}}} \frac{\delta^2}{\delta \tilde{J}(\mathbf{r}) \delta \tilde{J}(0)} Z_J^{\text{SG}} \right]_{J=0, \Phi=0} \\ &= \frac{1}{2\pi} V_L(\mathbf{r}) = h^{-1}(\mathbf{r}, 0) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\mathbf{p} \cdot \mathbf{r}}}{p^2 + M^2} = \frac{1}{2\pi} K_0(Mr), \end{aligned} \quad (3.23)$$

which coincides with (2.38) except for the replacement of Δ_0 by M .

It is a little bit complicated to evaluate the point-charge density correlation function. But if the area of the 2D system tends to infinity, one eventually gets the same expression as (2.45) with $V_L(\mathbf{r}) = 2\pi G(\mathbf{r})$ defined in (3.23).

Although some formulas derived in the present method look superficially similar to those derived in the last section, the two methods do not lead to identical results because of the difference in spatial dimensions of the effective Hamiltonians to which the methods were, respectively, applied. Moreover, in the present method, calculational accuracy can be improved by expanding $\exp(-\epsilon S^{(2)})$ in (3.10) to successively higher orders in ϵ (cumulant expansion). In the following sections we only work to $O(\epsilon)$.

IV. THE CRITICAL LINE AND PHASE DIAGRAM

In what follows, unless otherwise stated, we shall confine ourselves to point-charge case, i.e., $f(\mathbf{p}) = 1$.

$$V_{\text{eff}}(\Phi) = \frac{1}{2}D - \frac{1}{2}M^2 h^{-1}(\mathbf{r}, \mathbf{r}) - \alpha \beta^{-2} e^{-K \cos \beta \Phi}, \quad (4.1)$$

$$K = \frac{1}{2}\beta^2 \tilde{h}^{-1}(\mathbf{r}, \mathbf{r}). \quad (3.19a)$$

Now the mass-squared parameter M^2 can be fixed via the variational condition

$$\frac{\partial V_{\text{eff}}}{\partial M^2} = 0. \quad (3.20)$$

For point-charge case $f(\mathbf{r}) = \delta(\mathbf{r})$, one easily finds from (3.20) that

$$M^2 = \frac{2z\beta^2}{\xi^2} e^{-K \cos \beta \Phi}. \quad (3.21)$$

The stability criterion

$$\frac{\partial^2 V_{\text{eff}}}{\partial (M^2)^2} = \frac{1}{4\pi M^2} \left[1 - \frac{\beta^2}{8\pi} \right] \geq 0 \quad (3.22)$$

leads to the same critical temperature $T_c = \frac{1}{4}$ as that in Gaussian wave functional method.

The one-particle Green's function in the $f(p) = 1$ and $\Phi = 0$ case reads

where, as before, $\alpha = 2z\beta^2/\xi^2 = 4\pi z/\xi^2 T$. Then at the classical level, Eq. (2.1), the two parameters α and β can be expressed as

$$\alpha = \frac{d^2 V}{d\phi^2} \bigg|_{\phi=0}, \quad -\alpha\beta^2 = \frac{d^4 V}{d\phi^4} \bigg|_{\phi=0}. \quad (4.2)$$

We define the renormalized version of those parameters

$$\alpha_R \equiv \frac{d^2 V_{\text{eff}}}{d\Phi^2} \bigg|_{\Phi=0}, \quad -\alpha_R \beta_R^2 \equiv \frac{d^4 V_{\text{eff}}}{d\Phi^4} \bigg|_{\Phi=0} \quad (4.3)$$

to be those obtained with the appropriate ultraviolet and infrared cutoffs applied to all momentum integrations, as explained in Sec. II. For instance, the integral D defined in (3.17) is

$$\begin{aligned} D &= \frac{1}{2\pi} \int_{\lambda_c^{-1}}^{\xi^{-1}} \rho dp \ln(p^2 + M^2) \\ &= \frac{1}{2\pi} \left[\frac{\xi^{-2}}{2} \ln(\xi^{-2} + M^2) - \frac{\lambda_c^{-2}}{2} \ln(\lambda_c^{-2} + M^2) \right. \\ &\quad \left. - \frac{1}{2}(\xi^{-2} - \lambda_c^{-2}) + \frac{M^2}{2} \ln \frac{\xi^{-2} + M^2}{\lambda_c^{-2} + M^2} \right], \end{aligned} \quad (4.4)$$

$$h^{-1}(\mathbf{r}, \mathbf{r}) = \frac{\partial}{\partial M^2} D = \frac{1}{4\pi} \ln \frac{\xi^{-2} + M^2}{\lambda_c^{-2} + M^2}, \quad (4.5)$$

$$K = \frac{\beta^2}{8\pi} \ln \frac{\xi^{-2} + M^2}{\lambda_c^{-2} + M^2}. \quad (4.6)$$

The self-consistent condition Eq. (3.21) implies that

$$\frac{dM^2}{d\Phi} = \beta \alpha e^{-K \sin \beta \Phi} \left[\frac{\beta M^2}{8\pi} \left(\frac{1}{\lambda_c^{-2} + M^2} - \frac{1}{\xi^{-2} + M^2} \right) - 1 \right]^{-1}, \quad (4.7)$$

while the condition

$$\frac{dV_{\text{eff}}}{d\Phi} = 0 \quad (4.8)$$

leads to $\Phi = 0$, so that

$$\alpha_R = \alpha e^{-K} \equiv Z_\alpha \alpha, \quad (4.9)$$

which defines the renormalization constant Z_α . We now define the renormalized fugacity as $z_R = Z_\alpha z$, then from (3.21) and (4.9) we find

$$\eta_u \equiv M^2 \xi^2 = 4\pi z_R / T \\ = \frac{4\pi z}{T} [(1 + \eta_u)u / (1 + \eta_u u)]^{-1/4T}, \quad (4.10)$$

where $u \equiv \lambda_c^2 / \xi^2$ is the ratio of the infrared and ultraviolet lengths squared. In the physical limit $u \rightarrow \infty$, we have

$$\frac{4\pi z}{T} = (1 + \eta_\infty) [\eta_\infty / (1 + \eta_\infty)]^{(4T-1)/4T} \quad (4.11)$$

Now define the renormalized β by

$$\beta_R^2 \equiv Z_\beta \beta^2, \quad (4.12)$$

then after a lengthy calculation we obtain

$$Z_\beta = (A + 2B) / (A - B) \quad (4.13)$$

$$A = 8\pi [1 + \eta_u(1 + u) + \eta_u^2 u], \quad B = \beta^2 \eta_u (u - 1).$$

In the physical limit $u \rightarrow \infty$, $\eta_\infty u \gg 1$,

$$Z_\beta \rightarrow [8\pi(1 + \eta_\infty) + 2\beta^2] / [8\pi(1 + \eta_\infty) - \beta^2]. \quad (4.14)$$

This gives a condition for instability

$$1 + \eta_\infty = \frac{\beta_c'^2}{8\pi} \equiv \frac{1}{4T_c'} \quad (4.15)$$

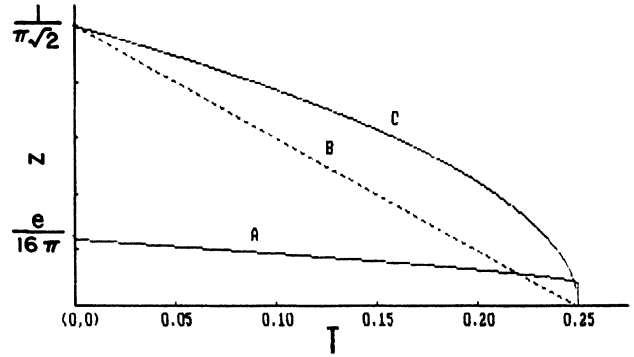


FIG. 1. The phase diagram of the two-dimensional Coulomb gas (point-charge case) in terms of temperature T and fugacity z ($\equiv e^{-E_c/T}$). The curve A refers to the border of validity for a sine-Gordon field theoretic approach, while B and C are the critical lines describing, respectively, the onset and completion of Kosterlitz-Thouless phase transition in this system. For more details, see text.

Inserting (4.15) into (4.11) then gives

$$\ln(16\pi z_c') = [1 - 1/(4T_c')] \ln(1 - 4T_c'). \quad (4.16)$$

Note that as η_∞ approaches zero (but still keeping $\eta_\infty u \gg 1$), (4.15) approaches Coleman's condition for phase transition, $\beta_c'^2 = 8\pi$, at which point $T_c' = \frac{1}{4}$. By carrying the procedure for renormalization^{7,17-19} twice, we have extended the region of validity of the SG field theory beyond the critical point, i.e., to a region where $\beta^2 > 8\pi$ ($T < \frac{1}{4}$).

The curve, or the critical line, given by (4.16) is shown as curve A in Fig. 1. Numerically, there are real values of η_∞ satisfying (4.11) in the area above curve A and restricted to $0 < T < \frac{1}{4}$, but no such values of η_∞ below curve A .

Now, curve A cannot be the critical line for the KT transition. For whereas a necessary condition for KT transition from the plasma (P) phase to the dipole (D) phase is that the correlation length $\lambda = 1/M$ approaches infinity, we have assumed the relation $u\eta_u = \lambda_c^{-2} M^2 \neq 0$ in deriving (4.11) and (4.16). Therefore, if one wishes to obtain the critical line for the KT transition, it is necessary to take an alternative approach.

Fortunately, there is a well-established expression for length-dependent dielectric function ϵ_r :^{8,2}

$$\frac{1}{\epsilon_r} = 1 + \frac{\pi^2}{T} \int_0^r dr' r'^3 \langle \Delta n(r') \Delta n(0) \rangle. \quad (4.17)$$

Using the same cutoff procedure as before and substituting $K_0(Mr)$ for $V_L(r)$ we obtain

$$\frac{1}{\epsilon_{\lambda_c}} \equiv \frac{1}{\epsilon_u} = 1 - \frac{2\pi^2 z^2}{T} \int_1^{\sqrt{u}} d\rho \rho^3 \left[1 - \exp \left(-\frac{2}{T} K_0(\rho \sqrt{\eta}) \right) \right] \exp \left[\frac{1}{T} [K_0(\rho \sqrt{\eta}) - K_0(\sqrt{\eta})] \right], \quad (4.18)$$

where $\rho = r/\xi$. We are now in a position to impose the $K-T$ transition condition $M \rightarrow 0$ on (4.18) by using the asymptotic expansion

$$\lim_{x \rightarrow 0} K_0(x) \rightarrow \ln \left[\frac{2}{x} \right] (1 + \frac{1}{4}x^2 + \dots) + [-\gamma + \frac{1}{4}(1-\gamma)x^2 + \dots], \quad (4.19)$$

where $\gamma = 0.5772$ is the Euler constant. Then it is easy to carry out the integration and get

$$\frac{1}{\epsilon_u} = 1 + \frac{2\pi^2 z^2}{1-4T} (u^{2-1/2T} - 1). \quad (4.20)$$

Taking the physical limit $u \rightarrow \infty$ one finds, for $T < \frac{1}{4}$

$$\frac{1}{\epsilon_\infty} = 1 - \frac{2\pi^2 z^2}{1-4T} \quad (T < \frac{1}{4}). \quad (4.21)$$

Since $\epsilon_\infty \equiv \bar{\epsilon}$ is just the dielectric constant of 2D CG, it seems that there are two alternatives to identify the critical line:

(i) Substitution of the condition

$$4\bar{\epsilon}T_c = 1 \quad (4.22)$$

into (4.21) leads to the critical line (curve *B* in Fig. 1)

$$z_c = \frac{1}{\pi\sqrt{2}} (1-4T_c). \quad (4.23)$$

(ii) Substitution of the condition

$$\bar{\epsilon} = \infty, \quad (T \geq \bar{T}_c) \quad (4.24)$$

into (4.21) gives the critical line (curve *C* in Fig. 1)

$$\bar{z}_c = \frac{1}{\pi\sqrt{2}} (1-4\bar{T}_c)^{1/2}. \quad (4.25)$$

Obviously one has $\bar{T}_c > T_c$ for the same value of E_c in *z*. We are now confronted with the question, which one is the real critical line? We propose the following resolution. Because condition (4.22) is derived from the competition between the potential energy and the entropy of a single CG particle,^{1,2} the corresponding critical line should be the upper bound of the pure *D* phase. In other words, curve *B* in Fig. 1 should delineate the onset of the

KT transition. Then curve *C* could delineate the completion of the KT transition from the *D* phase to the *P* phase, i.e., above curve *C* all dipoles are dissolved. Actually, it is the existence of a finite temperature range for evolution that characterizes the KT transition as a phase transition of infinite order.

The flow equation in the present scheme will be discussed in Appendix B.

V. THE ORDER PARAMETERS CHARACTERIZING THE KT TRANSITION

Having derived the explicit temperature dependence of correlation length, we are able to compute analytically a number of order parameters. We list several that are useful in discriminating the *P* phase from the *D* phase at the critical temperature T_c or $\bar{T}_c$.

(a) Correlation length (Debye screening length) $\lambda = 1/M$. As explained in the previous section, $M^2 = \eta_\infty \xi^{-2}$ is computed by solving (4.11). We obtain

$$\frac{1}{M} = \lambda = \begin{cases} \text{finite}, & T > T_c \\ \infty, & T < T_c. \end{cases} \quad (5.1)$$

(b) The density of free charge. First note that here the fugacity

$$z = e^{-E_c/T} = \exp\{[\mu + \frac{1}{2}U(0)]/T\},$$

with μ being the chemical potential and $\frac{1}{2}U(0)$ being the electrostatic self-energy of a particle in 2D CG². Thus the density of free charge can be expressed as

$$n(\mu, T) = \frac{N}{\Omega} = \frac{T}{\Omega} \frac{\partial}{\partial \mu} \ln Z_G, \quad (5.2)$$

where Ω is the area of the 2D system, and Z_G is the grand partition function that in the SG formulation is given by²

$$Z_G = \left\langle \exp \left[2z \int \frac{d^2 r}{\xi^2} \cos \beta \phi \right] \right\rangle_{\lambda_c^{-1}=0}. \quad (5.3)$$

Using the method of Gaussian effective potential described in Sec. III, one has

$$\frac{\partial}{\partial u} \ln Z_G = \frac{z}{T} \frac{\partial}{\partial z} \ln Z_G = \frac{Z}{T} \left\langle 2 \int \frac{d^2 r}{\xi^2} \cos \beta \phi \right\rangle_{\text{SG}} = \frac{2\Omega z}{\xi^2 T} e^{-\kappa \cos \beta \Phi} \quad (5.4)$$

consequently

$$n = \frac{TM^2}{2\pi} = \begin{cases} 0, & T < T_c \\ \text{finite}, & T > T_c. \end{cases} \quad (5.5)$$

(c) Charge number susceptibility $\chi(\mu, T)$. The charge number susceptibility $\chi(\mu, T)$ is analogous to the magnetic susceptibility χ in a magnetic system, and can be

defined as

$$\chi(\mu, T) = \frac{\partial n}{\partial \mu} = \frac{z}{T} \frac{\partial n}{\partial z}. \quad (5.6)$$

Using (5.5) and the results in Sec. III, one finds

$$\chi(\mu, T) = \frac{2T}{\pi} \frac{M^2}{(4T-1)} \begin{cases} = 0, & T < T_c \\ > 0, & T > T_c. \end{cases} \quad (5.7)$$

(d) Consider a charge with $\delta s = 1$ is put at the origin of 2D CG, then the free energy (precisely, the thermodynamical potential) of system becomes F_q . Let us calculate the difference between F_q and F_0 , which is the free energy in the case of absence of this charge,⁸ whereas

$$F_0 = -T \ln Z_G$$

$$= -T \ln \left\langle \exp \left[2z \int \frac{d^2 r}{\xi^2} \cos \beta \phi \right] \right\rangle_{\lambda_c^{-1}=0}, \quad (5.8)$$

this time we have

$$F_q = -T \ln Z'_G = -T \ln \left\langle \exp \left[2z \int \frac{dr}{\xi^2} \cos \beta \phi \right] e^{i\delta s \beta \phi(0)} \right\rangle_{\lambda_c^{-1}=0}. \quad (5.9)$$

Hence

$$F_q - F_0 = -T \ln \langle e^{i\delta s \beta \phi(0)} \rangle_{SG} = -T \ln (e^{-K} e^{i\delta s \beta \Phi(0)}) = TK = \frac{1}{4} \ln \frac{\xi^{-2} + M^2}{\lambda_c^{-2} + M^2} = \frac{1}{4} \ln \left[1 + \frac{1}{\eta_\infty} \right], \quad (5.10)$$

where (4.6) and $\eta_\infty = \lim_{\lambda_c \rightarrow \infty} M^2(u) \xi^2$ in (4.11) have been used. Therefore

$$F_q - F_0 = \begin{cases} \infty, & t < \bar{T}_c \\ \text{finite}, & T > \bar{T}_c \end{cases} \quad (5.11)$$

could also be served as an order parameter characterizing the KT transition.

(e) The dielectric constant

$$\bar{\epsilon} = \begin{cases} 1 + \frac{2\pi^2 z^2}{1 - 4T - 2\pi^2 z^2}, & (T < \bar{T}_c) \\ \infty, & ((T > \bar{T}_c)). \end{cases} \quad (5.12)$$

Though all the above formulas are expressed in results of the Gaussian effective potential (GEP) method, they remain valid in the Gaussian wave functional (GWF) method with the only replacement of Δ_0^2 after M^2 .

VI. SUMMARY AND DISCUSSION

Based on the SG field theory formulation of 2D CG, we have tried to propose a nonperturbative method, either GWF or GEP, for explicit calculations. Being a variational method essentially, as reflected in some self-consistent relation like (2.30) or (4.7), this method has both advantages and disadvantages. It systematically adds up the contributions of a large number of Feynman diagrams, with the loop number running from the lowest to infinity. In the research of relativistic quantum field theory, this kind of approach had been given different names by different authors, e.g., the Gaussian approximation,²⁰ optimized expansion method,¹² superdaisy approximation,²¹ or the cactus approximation.²² In detailed cases a comparison can be made between the GWF (or GEP) method and exact solutions,^{11,12} the GWF (or GEP) method does exhibit its priority over the usual perturbative approach, not only in accuracy but also in less expense of labor. We believe that the GEP (or GWF)

method is particularly accurate in dealing with the SG system, as discussed in Refs. 10 and 14; this is why we intend to study it in this paper. Of course, one still can not estimate the accuracy of a GEP calculation. Moreover, sometimes one is not sure if the symmetry in an original Lagrangian survives as a whole after a variational procedure has been made, whereas a perturbative loop expansion can do this order by order. But we believe that one is in a more secure situation as far as a real scalar field is concerned. We hope to apply this method to other problems in statistical physics, especially when trying to deal with some simplified model where an exact solution is known. (A simple model, related to this paper, is proposed to simulate the deconfinement of quark matter.)²³

Let us go back from general consideration to the 2D CG system discussed in this paper, many problems remain to be answered. First of all, if one leaves the ideal point-charge case and considers the charge as having finite spatial extension, $f(\mathbf{p}) \neq 1$, then a new length scale will enter the theory and the ultraviolet cutoff ξ^{-1} will no longer be needed. We believe that all the results presented earlier would be changed quantitatively even qualitatively.

ACKNOWLEDGMENTS

This work was supported in part by grants from the National Science Foundation of China, and the Natural Sciences & Engineering Research Council of Canada. We thank Professor X. Sun for helpful discussions. Ni and Chen wish to thank Dr. J. C. D. Milton for the kind invitation and warm hospitality at Chalk River where this paper was completed.

APPENDIX A: THE MATHEMATICAL TRICK IN THE GWF METHOD

Denoting

$$|\Psi\rangle_J = N_g \exp \left[\frac{i}{h} \int_x \mathcal{P}_x \phi_x - \frac{1}{2\hbar} \int_{xy} (\phi_x - \Phi_x) g_{xy} (\phi_y - \Phi_y) + \frac{1}{2} \int_x J_x \tilde{\phi}_x \right], \quad (A1)$$

where $\int_x = \int dx$ and an external source $J_x = J(x)$ is introduced. The normalization condition $\langle \Psi | \Psi \rangle_{J=0} = 1$ leads to

$$\langle \Psi | \Psi \rangle_J = \exp \left[\int_x J(x) \tilde{\Phi}(x) + \frac{\hbar}{4} \int_{xy} J(x) \tilde{g}_{xy}^{-1} J(y) \right] = \exp \left[\int_x \tilde{J}(x) \Phi(x) + \frac{\hbar}{4} \int_{xy} \tilde{J}(x) \tilde{g}_{xy}^{-1} \tilde{J}(y) \right], \quad (\text{A2})$$

where

$$\tilde{J}(x) = \int J(x') f_{\xi}(x' - x) dx', \quad \int_x J_x \tilde{\Phi}_x = \int_x \tilde{J}_x \Phi_x. \quad (\text{A3})$$

For example, one can get

$$\langle \Psi | \phi_x | \Psi \rangle = \left[\frac{\delta}{\delta \tilde{J}_x} \langle \Psi | \Psi \rangle_J \right]_{J=0} = \Phi_x, \quad (\text{A4})$$

$$\langle \Psi | \phi_x \partial_x^2 \phi_x | \Psi \rangle = \left[\frac{\delta}{\delta \tilde{J}_x} \partial_x^2 \frac{\delta}{\delta \tilde{J}_x} \langle \Psi | \Psi \rangle_J \right]_{J=0} = \Phi_x \partial_x^2 \Phi_x + \frac{\hbar}{2} \int_y \delta(x - y) \partial_x^2 \tilde{g}_{xy}^{-1}, \quad (\text{A5})$$

$$\langle \Psi | \pi_{\phi_x} | \Psi \rangle = \left\langle \Psi \left| -i\hbar \frac{\delta}{\delta \phi_x} \right| \Psi \right\rangle_{J=0} = \mathcal{P}_x, \quad (\text{A6})$$

$$\langle \Psi | \pi_{\phi_x}^2 | \Psi \rangle = \left\langle \Psi \left| \left[-i\hbar \frac{\delta}{\delta \phi_x} \right]^2 \right| \Psi \right\rangle_{J=0} = \mathcal{P}_x^2 + \frac{\hbar}{2} g_{xx}, \quad (\text{A7})$$

$$\langle \Psi | e^{i\beta \tilde{\phi}_x} | \Psi \rangle = \langle \Psi | \Psi \rangle_{J_y = i\beta \delta(y-x)} = \exp \left[-\frac{\hbar}{4} \beta^2 \tilde{g}_{xx}^{-1} \right] \exp(i\beta \tilde{\Phi}_x), \quad (\text{A8})$$

$$\langle \Psi | \exp[i\beta(\tilde{\phi}_x \pm \tilde{\phi}_0)] | \Psi \rangle = \langle \Psi | \Psi \rangle_{J_y = i\beta[\delta(y-x) \pm \delta(y)]} = \exp \left[i\beta(\tilde{\Phi}_x \pm \tilde{\Phi}_0) - \frac{\hbar \beta^2}{2} (\tilde{g}_{00}^{-1} \pm \tilde{g}_{x0}^{-1}) \right], \quad (\text{A9})$$

where the fact $\tilde{g}_{x0}^{-1} = \tilde{g}_{0x}^{-1}$ and $\tilde{g}_{00}^{-1} = \tilde{g}_{xx}^{-1}$ had been used.

APPENDIX B: THE FLOW EQUATION IN GEP METHOD

Starting from the length-dependent dielectric function (4.20) and defining a running “temperature” T_R by

$$\frac{1}{T_R} \equiv \frac{1}{T\epsilon_u} \quad (\text{B1})$$

one easily gets a flow equation for $1/T_R$ as follows:

$$\frac{d}{dl} \left[\frac{1}{T_R} \right] = u \frac{d}{du} \frac{1}{T_R} = \left[4 - \frac{1}{T} \right] \left[\frac{1}{T_R} - \frac{1}{T} \right] - \frac{2\pi^2 z^2}{T^2}, \quad (\text{B2})$$

where a new running variable $l = \ln u = 2 \ln \lambda_c / \xi$ is introduced. Equation (B2) can be integrated with respect to l (from $l=0$ to $l=\infty$) straightforwardly and reproduces the result (4.21) as expected. Notice that the running variable u may be expressed in terms of T_R and T via (4.20) and (B1):

$$u = \left[1 - \frac{T(4T-1)}{2\pi^2 z^2} \left[\frac{1}{T_R} - \frac{1}{T} \right] \right]^{2T/(4T-1)} \quad (\text{B3})$$

On the other hand, Eq. (4.10), i.e.,

$$\frac{4\pi}{T} z_R = \frac{4\pi z}{T} \left[\frac{(1+4\pi z_R/T)u}{1+4\pi z_R u/T} \right]^{-1/4T} \quad (\text{B4})$$

will give us the second flow equation as

$$\begin{aligned} \frac{dz_R}{dl} &= u \frac{d}{du} z_R \\ &= \frac{-z_R(1+4\pi z_R/T)/(2T)}{(1+4\pi z_R/T)(1+4\pi z_R u/T) + \pi^2 z_R(1-u)/T^2}. \end{aligned} \quad (\text{B5})$$

Substituting (B3) into (B5) will recast the latter into a form as

$$\frac{dz_R}{dl} = F(z_R, T_R, z, T), \quad (\text{B6})$$

with the function F independent of u as desired. Note that, however, whereas the second flow equation (B6) depends on T_R , the first one (B2) does not depend on z_R . The point (T_R, z_R) on phase diagram runs from its initial value (T, z) at $u=1$ to a fixed point $(T_R^* = T\epsilon, z_R^*)$ at $u \rightarrow \infty$.

Obviously the flow equations here with their results are different from that of renormalization-group (RG) method.^{8,9,24} Especially, our critical line B is a straight line with slope

$$dz_c/dT_c = -4/\pi\sqrt{2} = -0.9003,$$

while the slope of critical line curve at $z \rightarrow 0$ in the RG method²⁴ reads

$$\left. \frac{dz(l)}{dT(l)} \right|_{l \rightarrow \infty} \xrightarrow{T \rightarrow 1/4, z \rightarrow 0} -\frac{2}{\pi} = -0.6366.$$

Moreover, we did not find any special point discriminating two distinct transitions on the critical trajectory.²⁴ As our critical line is derived from Eq. (4.20), which only depends on (4.17) and (4.18) together with the condition

$M \rightarrow 0$, we do not know the reason for the discrepancy between these two approaches at this moment, so further investigation is needed.

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