DOUBLY RADIATIVE np CAPTURE AND THE ELECTRIC DIPOLE OPERATOR

H.C. LEE and F.C. KHANNA
Atomic Energy of Canada Limited, Chalk River Nuclear Laboratories, Chalk River, Ontario, Canada, K0J 1J0

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The (E1, E1) doubly radiative np capture cross section is calculated using several methods. Upper and lower limits are obtained. Significant discrepancies in the results obtained using respectively the correct dipole operator and the approximate operator $e\nabla\cdot\varepsilon$ are critically analyzed.

A great deal of attention has been given to the process of doubly radiative np capture, $n + p \to d + \gamma + \gamma$, after the reported measurement of a very large (350 μb) cross section by Dress et al. [1]. Although subsequent measurements [2, 3] and Monte Carlo calculations [4] have shown the earlier measurements to be instrumental[‡1], it is still important that the cross section for two photon emission, $\sigma_{2\gamma}$, be correctly calculated. Several calculations for $\sigma_{2\gamma}$ have been reported [6–9, 16]. Other than the so-called non-orthogonality term proposed by Adler [6], which will not be considered here, the results can be represented[‡2] by those of Blomqvist and Ericson [8] (BE) and Hyuga and Gari [9] (HG). BE showed that the leading process is the emission of two E1 photons and reported a value of 0.12 μb for $\sigma_{2\gamma}$. HG reported a value of 8.3 $\times 10^{-2}$ μb. The close agreement between these two results appear to be even more significant since BE used the "gradient operator" $e\nabla\cdot\varepsilon$ as the electric dipole interaction whereas HG used the long wavelength approximated dipole operator $e[H, \boldsymbol{r}\cdot\varepsilon]$, where ε is the polarization vector of the photon. Here we point out that this agreement is misleading. We show, by various calculations, that the result of HG is an upper limit, that the correct dipole operator (without the long wavelength approximation) gives results which are significantly smaller than that of BE, and that the difference is due to the fact that the gradient operator is biased toward high-energy contributions from the intermediate states.

The gradient operator suffers from two related defects: it is not gauge invariant and $e\nabla$ represents a non-conserved current. For the emission (or absorption) of low energy photons from (or by) a bound state, the difficulty associated with specifying a conserved current, $\boldsymbol{J}$, is removed by the well-known Siegert theorem [10, 11]. This theorem was extended to photons of any energy and for unbound states by Sachs and Austern [12] and by Foldy [13]. Suffice it to say that in this theorem, by making use of the continuity equation, the operator $\boldsymbol{J}\cdot\varepsilon$ (to which the gradient operator is an approximation) is replaced by the gauge invariant dipole operator which, in relative coordinates, is[‡3]

$$E1(\omega) = \frac{3e}{\omega}\left[H, j_1\left(\frac{\omega r}{2}\right)\hat{\boldsymbol{r}}\cdot\varepsilon\right], \tag{1}$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ and ω is the photon energy (we use the units $\hbar = c = M = 1$, where M is the nucleon mass). We make a final remark before computing $\sigma_{2\gamma}$ using this operator. Grechukhin [14] and HG have shown that the matrix element for the emission of two electric photons (with energies ω_1 and ω_2) is proportional to $\omega_1\omega_2$ rather than to $(\omega_\mathrm{f} - \omega_n)(\omega_n - \omega_\mathrm{i})$ as one would naively expect from (1). The indices f, n, and i subscribe respectively to the final, intermediate and initial states. Therefore effectively the dipole operator becomes $3e\,j_1(\omega r/2)\,\boldsymbol{r}\cdot\varepsilon$, which is the form we shall use.

The differential cross section for two photon emission is

[‡1] Alburger [5] first suggested that the observed events could be due to annihilation of positron in flight.

[‡2] The calculation by Grechukhin [7] is similar to that by BE, except that a factor of 8 was missing in the result, as was pointed out by the latter authors.

[‡3] Later we show contribution from terms proportional to $(1/\omega)\,j_\lambda(\omega r/2)$, $\lambda > 1$, can be neglected.

$$d\sigma_{2\gamma}/d\Omega = 3/(16\pi^2 v_n) \int_0^\omega d\omega_1 \omega_1 \omega_2 \sum_{q_1 q_2} |\langle M \rangle|^2 \qquad (2)$$

where q_1, q_2 are the polarizations of the two photons, v_n is the neutron velocity, $\omega = 2.2$ MeV is the deuteron binding energy, $\omega_1 + \omega_2 = \omega$, and

$$\langle M \rangle = \sum_n \left\{ \langle \psi_d | E1(\omega_1) | \psi_n \rangle \frac{1}{\omega_n + \omega_2} \langle \psi_n | E1(\omega_2) | \psi_t \rangle \right. \qquad (3)$$
$$\left. + (1 \rightleftarrows 2) \right.$$

where ψ_d, ψ_n and ψ_t are the relative wavefunctions of the np system in the deuteron, intermediate and unbound triplet ($^3S_1$) states. In the following we will do three types of calculations using both the dipole and gradient operators: (i) a calculation where the intermediate states are taken to be plane waves; (ii) where $\langle M \rangle$ is expressed in terms of various empirical energy weighted sum rules of the deuteron photo-disintegration cross section, $\sigma_{\gamma d}$, and (iii) where the closure approximation is used. For ψ_d and ψ_t we adopt the wavefunctions used by BE: $\psi_d(r) = \sqrt{\kappa/2\pi}\, e^{-\kappa r}/r$ and $\psi_t(r) = \sin(kr + \delta_t)/kr$, where $\kappa = \sqrt{\omega}$ and $\tan \delta_t = -ka_t$, a_t (= 5.4 fm) being the triplet scattering length.

(i) Plane wave intermediate states. In this case $\psi_n(r) = \exp(i\boldsymbol{p}\cdot\boldsymbol{r})$, $\Sigma_n \to (2\pi)^{-3} \int d^3p$ and $\omega_n = p^2$, in eq. (3). The result for the gradient operator was given by BE,

$$\sigma_{2\gamma} = \bar{\sigma}_{2\gamma}(20 + \pi - 32 \ln 2) = 0.12\ \mu b, \text{ (gradient)} \qquad (4)$$

where $\bar{\sigma}_{2\gamma} = \frac{2}{9}\alpha^2 v_n^{-1} a_t^2 \omega^{5/2}$. The result for the dipole operator is

$$\sigma_{2\gamma} = \bar{\sigma}_{2\gamma} \int_0^1 d\gamma\, \gamma^3 (1-\gamma)^3 \left[\frac{2+\sqrt{\gamma}}{(1+\sqrt{\gamma})^2 \sqrt{\gamma}} + \frac{2+\sqrt{1-\gamma}}{(1+\sqrt{1-\gamma})^2 \sqrt{1-\gamma}} - \frac{3}{a_t \kappa} \frac{1}{\gamma(1-\gamma)} \right]^2 \qquad (5)$$
$$= 6.9 \times 10^{-2}\ \mu b. \text{ (dipole)}$$

The latter result is smaller by a factor of 16. In the following we show that this result is not due to accidental cancellation but is due to real physical effects.

(ii) Sum rule. We may relate the dipole matrix elements in (3) to that of the photodisintegration of the deuteron. For the dipole operator

$$\sigma_{2\gamma} = \frac{2}{(2\pi)^5} \frac{1}{v_n} \int_0^\omega d\omega_1 \omega_1^3 \omega_2^3 \left| \int_\omega^\infty d\omega_\gamma R(\omega_\gamma) \times \left(\frac{1}{\omega_\gamma - \omega_1} + \frac{1}{\omega_\gamma - \omega_2} \right) \frac{\sigma_{\gamma d}(\omega_\gamma)}{\omega_\gamma} \right|^2 \text{ (dipole)} \qquad (6)$$

where $\omega_\gamma = p^2 + \omega$ is the photon energy in $\gamma + d \to n + p$ and[‡4] $R = \omega_2 \langle \psi_t | j_1(\omega_1 r/2) \hat{r} | \psi_n \rangle / \omega_1 \langle \psi_d | j_1(\omega_2 r/2) \hat{r} | \psi_n \rangle$. For the gradient operator, we have

$$\sigma_{2\gamma} = \frac{2}{(2\pi)^5} \frac{1}{v_n} \int_0^\omega d\omega_1 \omega_1 \omega_2 \left| \int_\omega^\infty d\omega_\gamma R'(\omega_\gamma) \times \left(\frac{1}{\omega_\gamma - \omega_1} + \frac{1}{\omega_\gamma - \omega_2} \right) \omega_\gamma \sigma_{\gamma d}(\omega_\gamma) \right|^2 \text{ (gradient)} \qquad (7)$$

where $R' = \langle \psi_t | \mathbf{v} | \psi_n \rangle / \langle \psi_d | \mathbf{v} | \psi_n \rangle$. The two expressions (6) and (7) are qualitatively different. The high-energy region contributes more strongly in the case of the gradient operator. To evaluate these equations, we use the inequalities

$$\frac{1}{\omega_1} + \frac{1}{\omega_2} \geq \frac{1}{\omega_\gamma - \omega_1} + \frac{1}{\omega_\gamma - \omega_2} \geq \frac{2}{\omega_\gamma}. \qquad (8)$$

Since for any reasonable intermediate wavefunctions R and R' are positive definite, this inequality can be used outside the integral sign. If we now evaluate the averages $\langle R \rangle_m = \int_\omega^\infty R\, \omega_\gamma^m \sigma_{\gamma d}\, d\omega_\gamma / \sigma_m$, $\sigma_m = \int_\omega^\infty \omega_\gamma^m \sigma_{rd}\, d\omega_\gamma$ using plane waves, approximate bounds for $\sigma_{2\gamma}$ can be expressed in terms of empirical sum rules. These are [15] [‡5] 41 mb-MeV, 3.9 mb and 0.7 mb/MeV for σ_0, σ_{-1} and σ_{-2} respectively.

Putting all pieces together we have, for the dipole operator

$$1.3 \times 10^{-1}\ \mu b > \sigma_{2\gamma} > 6.9 \times 10^{-2}\ \mu b. \text{ (dipole)} \qquad (9)$$

For the gradient operator the upper bound diverges and we have only a lower bound

$$\sigma_{2\gamma} > 2/(3\pi^4\alpha^2)\, \bar{\sigma}_{2\gamma} \sigma_0^2 = 1.6 \times 10^{-1}\ \mu b. \text{ (gradient)} \qquad (10)$$

‡4 In the region of interest, the dipole matrix element is effectively proportional to ω_1 (or ω_2), even for the two unbound states ψ_t and ψ_n.

‡5 The value of 0.045 mb/MeV for σ_{-2} quoted in ref. [15] is incorrect. The value used here is evaluated from $\sigma_{\gamma d}$ data. Also for all sum rules the high energy was cut off at the pion production threshold.

Table 1
Comparison of results of dipole and gradient operators

Type of calculation	Dipole operator	Gradient operator
i) plane wave int. state	$\sigma_{2\gamma}^{a)} = 6.9 \times 10^{-2}$	$\sigma_{2\gamma} = 0.12$
ii) sum rule	$1.3 \times 10^{-1} > \sigma_{2\gamma} > 6.8 \times 10^{-2}$	$\sigma_{2\gamma} > 0.16$ b)
iii) closure	$\sigma_{2\gamma} \approx 4.3 \times 10^{-2}$	diverges c) at $r = 0$

a) Cross section in units of μb; b) See text for apparent contradiction with (i); c) See text for detail.

Note that from (8) the second inequality can in fact be viewed as a very good approximation. Therefore we have quantitative agreement between results obtained here (lower bound in (9)) and those in the last section. The fact that the result in (10) is greater than the result of BE is not self-contradictory because the empirical σ_0 is greater than that calculated with plane wave (~30 mb-MeV).

(iii) Closure approximation. In eq. (3), if we replace the energy denominators by an average value then we have the closure approximation. For the dipole operator

$$\sigma_{2\gamma} = \frac{1}{144\pi}\frac{\alpha^2}{v_n}\int_0^{\omega} d\omega_1 \omega_1^3 \omega_2^3 \left(\frac{1}{\omega_n + \omega_1} + \frac{1}{\omega_n + \omega_2}\right)^2 \times |\langle\psi_d|r^2|\psi_t\rangle|^2, \tag{11}$$

where the long wavelength approximation has been used because ψ_d is bound. It is here that we justify ignoring the higher multipoles. Again we can make use of the inequality $1/(\omega_n + \omega_1) + 1/(\omega_n + \omega_2) \leqslant \omega/\omega_1\omega_2$ to get an estimate

$$\sigma_{2\gamma} \approx \tfrac{1}{6}\bar{\sigma}_{2\gamma}(3/a_t\kappa - 1)^2 = 4.3 \times 10^{-2}\ \mu\text{b}. \tag{12}$$

This result is analogous to that of HG, who used slightly different wavefunctions for ψ_d and ψ_t. estimate. Another estimation is obtained by setting $\langle\omega_n\rangle = \omega$, a value which corresponds to the maximum value of $\sigma_{\gamma d}$. We then have

$$\sigma_{2\gamma} \approx \bar{\sigma}_{2\gamma}(3/a_t\kappa - 1)^2[184/3(\ln 2) - 85/2] = 3.4 \times 10^{-3}\ \mu\text{b}. \tag{13}$$

A similar calculation using the gradient operator leads to difficulties because the integral $\int(\nabla\psi_d)\cdot(\nabla\psi_t)\,d^3r$ diverges as $r \to 0$, unless short-range correlations between nucleons are taken into account. This is not at all surprising since we have already seen that the gradient operator is biased towards high-energy components.

The results obtained in this letter are summarized in table 1. We have shown that the dipole operator, which implies a conserved current and is therefore gauge invariant, is not sensitive to properties of the nucleus at very high energies or very short distances, properties which we know little about. The gradient operator has the exact opposite characteristic and also gives a value for $\sigma_{2\gamma}$ which is too large.

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