THE LIGHT-CONE GAUGE AND FINITENESS OF $N=4$ SUPERSYMMETRIC THEORY*

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Received 4 November 1986

Non-abelian gauge theories in the light-cone gauge (defined by the Mandelstam–Leibbrandt prescription) can have non-local counterterms Nevertheless, the possible counter-terms are severely restricted by gauge-invariance We detail the possible form of the counter-terms This enables us to complete the proof of the finiteness of $N=4$ SUSY, by showing that if all three- and higher-field counter-terms are zero, then the two-field ones are also zero

1 Introduction It has been shown [1,2] that all three-point and higher functions are finite in the $N=4$ supersymmetric model (SUSY) in the light-cone gauge superfield formalism To complete the proof of finiteness, we must show that the two-point functions are also finite This is expected to follow in some way from Ward identities, but in the light-cone formalism of the light-cone gauge [3] (called LC2 in ref [4]) there do not appear to be any Ward identities, because there are no unphysical degrees of freedom in the fields

One way to proceed is indirectly, using the four-component form of the light-cone gauge (called LC4 in ref [4]) as an intermediate step But, even in LC4, the Ward identities work in an unusual way, since infinite terms can be non-local in the Mandelstam–Leibbrandt prescription for the light-cone gauge [1,5]

In section 2, we examine the possible forms of the counter term in pure Yang–Mills theory in LC4, using the constraints of gauge-invariance and Lorentz-invariance The light-cone gauge turns out to be peculiar Since it is a δ-function gauge, it is ghost-free, and simple QED-type Ward identities are applicable [6] In this respect, it is like the axial gauge But, in LC4, counter-terms are in general non-local [4–9], so that counter-terms are permitted which are not *explicitly* Lorentz invariant We detail the constraints on them, and list the possibilities

In section 3, we give the consequences of our analysis for $N=4$ SUSY It turns out that the possible counter-terms have the property that, when reduced to LC2, the two-field ones vanish if the three-field ones do This allows the completion of the proof of the finiteness of $N=4$ SUSY

2 Counter-terms in LC4 A general analysis of the counter-terms in LC4 was given in ref [9] As in any gauge, they have the form

$$\Gamma = Y + \Delta X \tag{1}$$

where Δ is the BRS operator [10]

$$\Delta = \frac{\delta S}{\delta A} \cdot \frac{\delta}{\delta J} + \frac{\delta S}{\delta J} \cdot \frac{\delta}{\delta A} + \;, \tag{2}$$

X is any functional of A, J, ω, K (respectively, the gauge field, the source associated with A, the ghost field and the source associated with ω), and Y is an explicitly gauge invariant function of A, i e

$$\frac{\delta S}{\delta J} \cdot \frac{\delta Y}{\delta A} = 0 \tag{3}$$

In ref [9], counter-terms to one-loop order were considered But, for the purposes of studying $N=4$ SUSY, we may suppose we are examining counter-

* Supported in part by a NATO Collaborative Research Grant

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term to the *first* non-vanishing order l in the loop expansion (if any) Then S is the un-renormalized action, and is unchanged to $(l-1)$ loops, by hypothesis

In general, ΔX in (1) will produce ghst-terms But ghosts do not interact in the light-cone gauge, since their vertex with a gluon is proportional to n_μ (the defining light-like vector), and the propagator is a projection operator perpendicular to n_μ It follows that, in this case, X must be independent of ω and K, and in fact have the form

$$X=J_\mu \cdot M_\mu(A) \tag{4}$$

(This observation appears to contradict eq (14) and (11) of ref [9], which appears to contain ghost terms But in fact the ghost terms cancel out when their equations $a_4=0$, $a_5=a_7$ are used This is, of course, no accident)

The trivial nature of (4) was implicitly recognized in refs [4,6,8], in verifying the naive (i e ghostless) Ward identity

$$q_\nu \Gamma^{abc}_{\nu\mu\sigma}=\delta^{abc}[\Pi_{\mu\sigma}(r)-\Pi_{\mu\sigma}(p)] \tag{5}$$

In any gauge except the light-cone gauge, dimensionality restricts M_λ to be a linear function of A, and we would have

$$M_\lambda=a_1A_\lambda+a_2n_\lambda n\cdot A \tag{6}$$

But, in the light-cone gauge [4–9], power-counting is relaxes to the extent that the divergent terms may contain non-localities involving $(n\cdot\partial)^{-1}$ (but not other non-localities) The Mandelstam–Leibbrandt prescription is also peculiar in that it entails in its definition *another* light-like vector n^*, which we may choose to satisfy $n\cdot n^*=1$ So M may depend upon n^* as well as n Lastly, there is invariance under Lorentz boosts, giving

$$n\rightarrow e^{\theta}n, \quad n^*\rightarrow e^{-\theta}n^* \tag{7}$$

This has the consequence that the number of n and n^* must balance As an example, the second term in(6) would not be allowed in the light-cone gauge but

$$M_\lambda=a_1A_\lambda+a_2n_\lambda n^*\cdot A+a_3n^*_\lambda n\cdot A \tag{8}$$

would be allowed

There is a further restriction upon $M(A)$, which comes from the fact that

$$\Delta X=\frac{\delta S}{\delta J}\cdot\left(J\frac{\delta M}{\delta A}\right)+\frac{\delta S}{\delta A}\cdot M$$

should be ghost-less This requires that

$$\frac{\delta S}{\delta J_\lambda}\cdot\frac{\delta M_\mu}{\delta A_\lambda}=D_\lambda\omega\cdot\frac{\delta M_\mu}{\delta A_\lambda}=0\,, \tag{9}$$

i e that M should be gauge-invariant This of course excludes all terms in (8), and indeed excludes any *local* terms in M_λ

We now write down some of the lowest terms in M_λ which *are* allowed by gauge-invariance, dimensionality, the invariance (7), and restricted non-locality These are

$$\begin{aligned}M_\lambda&=a_1(n\cdot D)^{-1}F_{\lambda\mu}n_\mu\\&\quad+a_2(n\cdot D)^{-1}F_{\nu\mu}n^*_\nu n_\mu n_\lambda\\&\quad+a_3(n\cdot D)^{-3}F_{\sigma\mu}n_\mu F_{\sigma\nu}n_\nu n_\lambda+\ \ ,\end{aligned} \tag{10}$$

where stands for terms similar to the last but with the factors of $(n\cdot D)$ distributed in different places

Note that, if there were m factors of F, dimensionality would require a factor $(n\cdot D)^{-2m+1}$ and so (7) would require at least $(2m-1)$ n-vectors in the numerator For $m\geqslant 3$, this becomes impossible because $n_\mu n_\nu F_{\mu\nu}\equiv 0$ Similar arguments apply if there is a D_λ in the numerator

Next, we must examine the possible terms in Y in (1) Y is an explicitly gauge-invariant scalar (whereas M_λ is an explicitly gauge-invariant vector) Possible terms in Y include

$$Y=b_1F_{\mu\nu}F_{\mu\sigma}+b_2n_\mu n^*_\nu F_{\mu\lambda}F_{\nu\lambda}+ \tag{11}$$

There might also, a priori, be non-local terms containing $(n\cdot D)^{-1}$ However, we now argue that Y (unlike X) must be *explicitly* Lorentz invariant, i e independent of n_μ and n^*_μ (The argument does not prevent the *finite parts* of vacuum polarization from having terms forbidden in counter-terms, indeed, a one-loop calculation [4] reveals that the finite part corresponding to the second term in (11) does not vanish)

For this argument, we use a result [11] which shows that, under an infinitesimal change of the gauge-fixing term (i e , in this case, of n and n^*) the change in the counter-term must have the form

$$\delta\Gamma=\Delta\Gamma'\,, \tag{12}$$

where Γ' is some functional of the fields, and Δ is the BRS operator (2) For the terms ΔX in (1), this is trivially satisfied by

$$\Gamma' = \delta X = J_\lambda \cdot \delta M_\lambda \tag{13}$$

But, for Y, we can show it entails a contradiction The argument follows

Suppose

$$\Gamma = Y \neq \Delta X$$

for any X Eq (12) gives

$$\Delta\Gamma' = \Delta W_\lambda \delta n_\lambda$$

or

$$\partial\Gamma/\partial n_\lambda = \Delta W_\lambda + \alpha n_\lambda + \beta n^*_\lambda ,$$

where α and β are lagrangian multipliers coming from the constraints $n^2=0$ and $n \cdot n^* = 1$ It follows that

$$\Delta(\partial W_\lambda/\partial n_\mu - \partial W_\mu/\partial n_\lambda) = 0 .$$

The solution of this is

$$W_\lambda = \partial X/\partial n_\lambda + T_\lambda ,$$

where $\Delta T_\lambda = 0$ Then

$$\partial\Gamma/\partial n_\lambda = \Delta\partial X/\partial n_\lambda + \alpha n_\lambda + \beta n^*_\lambda$$

$$= (\partial/\partial n_\lambda)(\Delta X) + \alpha n_\lambda + \beta n^*_\lambda$$

(since Δ is independent of n) or

$$\Gamma = \Delta X = \text{terms independent of } n$$

This contradicts the assumption, unless Y is independent of n, so terms like b_2 in (11) are excluded Thus the only possible term in Y is

$$Y = b_1 F_{\mu\nu} F_{\mu\nu} \tag{14}$$

We have now established the form of the counter-terms They are given by (1), (4), (10) and (14)

3 *The reduction to LC2* In order to make contact with what has been done in SUSY in the light-cone gauge, we must use LC2, since LC2 was used in the SUSY calculation [1,2] LC2 is defined by imposing the equations

$$n \cdot A = 0 , \quad n_\mu D_\sigma F_{\sigma\mu} = 0 , \tag{15,16}$$

which eliminate $n \cdot A$ and $n^* \cdot A$ in favour of a two-component vector A_i It is immediately seen that a_2 and a_3 and all similar terms in (10) do not contribute at all in LC2, because of (16) and because these terms are all proportional to n_λ Therefore, as far as LC2 is concerned, the counter-term is

$$\Gamma = b_1 F_{\mu\nu} \cdot F_{\mu\nu}$$

$$+ a_1 D_\sigma F_{\sigma\lambda} \cdot (n \cdot D)^{-1} F_{\lambda\mu} n_\mu \tag{17}$$

All that remains is to show that a_1 and b_1 contribute differently to the LC2 three- and four-point functions. In LC2, let us write

$$\Gamma|_{LC2} = b_1 I(A_i) + a_1 J(A_i) \tag{18}$$

Then one may verify, using (15) and (16), that

$$J = A_i \delta I/\delta A_i \tag{19}$$

It follows that the ratio of the three- and four-point terms in I and J is different The absence of three- and four-point counter-terms in LC2 therefore implies that

$$\Gamma|_{LC2} = 0 \tag{20}$$

This completes our proof that, in $N=4$ SUSY, the Yang–Mills sector of the counter-term is zero, if all three- and higher-point functions are finite [1,2] Then, by supersymmetry, the counter-term for the spinor sector must also be zero, ie, $N=4$ SUSY is finite

For completeness, we compare our form of the counter-terms with the results of refs [6,8,9], for pure Yang–Mills theory At one-loop order,

$$b_2 \sim \tfrac{11}{3}(n-4)^{-1} , \quad a_2 \sim 2(n-4)^{-1} , \quad a_1 = 0$$

Note that, although $a_2 \neq 0$, it is irrelevant to LC2 We suspect that there is some reason why $a_1 = 0$, but we do not know what it is Certainly the term in a_1 is consistent with gauge invariance

The authors acknowledge a helpful discussion with L Brink and J C T thanks CRNL for its hospitality during his visit there We are grateful to D R T Jones for telling us of an error in an earlier version of this paper

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