

A NEW METHOD FOR REGULATING FEYNMAN INTEGRALS IN THE AXIAL AND LIGHT-CONE GAUGES

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It is shown that the principle of analytic continuation is well suited for regulating the spurious singularity peculiar to Feynman integrals in the axial gauges. This new, analytic regularization is much more powerful than the standard regularization based on the principal-value prescription, at the same time yielding identical results compared to the latter. An analytic representation based on a Meijer G -function is derived for the class of massless, two-point integrals $\int d^2\omega q [(p-q)^2]^\kappa \times (q^2)^\mu (q \cdot n)^\nu$, where n is the vector defining the axial gauge and ω, κ, μ and ν are continuous variables. Several important aspects of the representation are discussed and examples of its application are given.

The axial gauge [1] is useful in the study of gauge theories and quantum gravity principally because in such a gauge Faddeev–Popov ghosts [2], required to satisfy Ward identities [3], are decoupled from other physical fields. The price one pays for this convenience is that Feynman integrals, in addition to having the usual ultraviolet (UV) and infrared (IR, for massless theories) singularities, may also contain singularities because the factor $q \cdot n$ appearing in the propagator for the massless vector boson,

$$\Delta_{\mu\nu}(q) \propto q^{-2} [\delta_{\mu\nu} - (q_\mu n_\nu + q_\nu n_\mu)/q \cdot n + n^2 q_\mu q_\nu / (q \cdot n)^2], \quad (1)$$

may vanish in the integrand; n is the vector defining the axial gauge condition $A \cdot n = 0$. Heretofore, the method that has been widely used to evaluate axial gauge integrals is the principal-value (PV) prescription [4], where the limiting process

$$(q \cdot n)^{-N} \rightarrow \frac{1}{2} \lim_{\eta \rightarrow 0} [(q \cdot n + i\eta)^{-N} + (q \cdot n - i\eta)^{-N}], \quad N \geq 2, \quad (2)$$

is used to regulate the singularity at $q \cdot n = 0$. Although the PV prescription has been demonstrated [1] to yield consistent results, its implementation is notoriously tedious and treacherous.

In this letter we describe a method that is more elegant and powerful than the PV prescription for regulating axial gauge singularities, and which regulates the IR and UV singularities simultaneously. The method is based on the principle of analytic continuation. It is similar to the rarely used analytic regularization of Bollini et al. [5], who employed it originally to regulate integrals in covariant gauges. Comparing the new method to the PV prescription is analogous to comparing dimensional regularization [6] to Pauli–Villars's method [7] of regulating UV divergences. Our method for regulating the Feynman integral is a hybrid of dimensional and analytic regularization, which has been used, tacitly, by others [1,6,8]. The method also enables us to regulate integrals in the light-cone gauge, constrained by $n^2 = 0$.

We now describe the method. First generalize the exponent N in (2) to a continuous variable ν and express the

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factor $(q \cdot n)^\nu$ using Euler's representation

$$(q \cdot n)^\nu = \frac{1}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} \exp[-(q \cdot n)t] dt. \quad (3)$$

The treatment may be extended to all factors in the Feynman integral. The dimensionality of (euclidean) space-time is then generalized to the continuous variable 2ω , just as in dimensional regularization. The resulting generalized Feynman integral S is a function of a set of continuous variables $(\omega, \nu, \dots)$. A regularization of S is obtained provided that: (i) there exists a finite region in the $(\omega, \nu, \dots)$ hyperspace within which S is well-defined, (ii) a function $\tilde{S}(\omega, \nu, \dots)$, can be found which is a representation of S within the region of (i) and which is well-defined exterior to the region. By the principle of analytic continuation the regulated S in the exterior region is equal to $\tilde{S}$. In other words, $\tilde{S}$ is the representation of S everywhere, and S in turn is the generalization of the "primal", but divergent integrals specified originally only at the point $(4, N, \dots)$. Consideration of "Feynman integrals" in this very general form leads to many other advantages in addition to the regularization of axial gauge singularities, these are discussed elsewhere [9]. It suffices to say that in the new method massless tadpole integrals are well-defined (they vanish), gauge invariance is preserved, there are no spurious γ_5 -anomalies, and derivatives with respect to exponents of the factors can be evaluated.

To be specific consider the class of integrals

$$S_{2\omega}(p, n, \omega, \kappa, \mu, \nu) \equiv \int d^2\omega q [(p-q)^2]^\kappa (q^2)^\mu (q \cdot n)^{2\nu+s}, \quad (4)$$

where ω, κ, μ, ν are continuous and $s = 0$ or 1 . In the limit $\omega \rightarrow 2, \{\kappa, \mu, \nu\} \rightarrow \text{integers } \{K, M, N\}$, the primal integrals $S_4(p, n; K, M, N, s)$ are two-point functions in massless theories. The integral (4) is well-defined in the region delineated by

$$\omega + \kappa + \mu + \nu < 0, \quad \omega + \kappa > 0, \quad \omega + \mu + \nu + s > 0, \quad \nu + s \geq -1/2, \quad (5)$$

where all variables are understood to be real. For example, in the vicinity of $\omega = 2$, the integral with $\{\kappa, \mu, \nu, s\} = \{-1, -1, -1, 1\}$ is regular. It can be shown [9] that in a region of this point the integral, for $|y| \equiv (p \cdot n)^2/(p^2 n^2)$, is represented by

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \frac{\pi \omega (p^2)^{\alpha_1} (n^2)^{\alpha_2} (p \cdot n)^s \Gamma(\alpha_2 + s + 1/2)}{\Gamma(\beta_1 - \alpha_1) \Gamma(\beta_1 - \alpha_1) \Gamma(-\alpha_0 - \alpha_1 - s) \Gamma(-\alpha_2)} G_{3,3}^{2,3} \left(y \left| \begin{matrix} 1 + \alpha_0, 1 + \alpha_1, 1 + \alpha_2 \\ 0, \beta_1; 1/2 - s \end{matrix} \right. \right), \quad (6)$$

where the "indices" are: $\alpha_0 = -(\omega + \mu + \nu + s)$, $\alpha_1 = \omega + \kappa + \mu + \nu$, $\alpha_2 = \nu$, $\beta_1 = \omega + \kappa + \nu$. The symbol G stands for a Meijer G -function [10] which can be represented as either a contour integral or as a sum of generalized hypergeometric functions. The G -function in (6) is a well-defined function for all values of its arguments. For $|y| > 1$, it has the analytic continuation [10]

$$G_{3,3}^{2,3}(y|\dots) + y^\nu G_{3,3}^{3,2} \left(\frac{1}{y} \left| \begin{matrix} 1 + \alpha_2, 1 + \alpha_2 - \beta_1; \alpha_2 + s + 1/2 \\ 0, \alpha_2 - \alpha_0, \alpha_2 - \alpha_1 \end{matrix} \right. \right). \quad (7)$$

The right-hand side of (6) is therefore well-defined for all values of its variables and by the principle of analytic continuation can be used to represent the integral in (4) exterior to (as well as within) the region given in (5). Primal integrals are defined by evaluating (5) as $\{\kappa, \mu, \nu\} \rightarrow \text{integers}$ with $\epsilon \equiv \omega - 2$ finite, and then allowing ϵ to approach zero. We now make several comments concerning (6).

(a). *The principal-value prescription.* It is shown in detail elsewhere [9] that our analytic method and the PV prescription yield identical results for the infinite and regular parts of the integral. A brief sketch of the proof is as follows. With the PV prescription, for ν equal to a negative integer $-N - s$, the right-hand side of (4) becomes

$$\lim_{\eta \rightarrow 0} R(\eta^2 \partial/\partial \eta^2) \int d^2\omega q [(p-q)^2]^\kappa (q^2)^\mu (q \cdot n)^s [(q \cdot n)^2 + \eta^2]^{-N-s}, \quad (8)$$

where

$$R(\eta^2 \partial^2 / \partial \eta^2) = \frac{\Gamma(-N+1/2)}{\Gamma(1/2)} \prod_{l=0}^{N-1} (l - N + 1/2 - \eta^2 \partial^2 / \partial \eta^2), \quad (9)$$

is a polynomial in the operator $\eta^2 \partial^2 / \partial \eta^2$ of order N . The integral in (8) can be represented by a contour integral which displays a string of singularities of leading order $(\eta^2)^{-N+1/2}$, all of which are precisely removed when the integral is operated upon by R . The result is a regular power series in η which, in the limit $\eta \rightarrow 0$, becomes exactly equal to the right-hand side of (6). In short, the PV prescription takes a circuitous path to the same result as the analytic method.

(b) *Gauge invariance.* It has been argued [11] that analytic regularization does not preserve gauge invariance. This argument actually only applies to the theory of Speer [12], where the method of analytic regularization is used, but not to the method [5] itself. In Speer's theory it is proposed that exponents of the denominator of all propagators in the theory, which normally have values of unity, be generalized to continuous variables. In contrast, in our method analytic regularization is employed only to regularize divergent integrals. The distinction between the two approaches is subtle but important. Since only primal integrals occur in perturbation theory, non-integral exponents in fact do not appear anywhere in the theory. Furthermore, since it is shown {see (a) and ref. [9]} that the values of primal integrals regularized with our method are identical to those with the PV prescription, which is known [1] to preserve gauge invariance, it follows that our method preserves gauge invariance also.

(c) *Covariant gauge.* When $\nu = s = 0$, (6) gives the known result [8]

$$d^2\omega q [(p-q)^2]^\kappa (q^2)^\mu = \frac{\pi^\omega (p^2)^{\omega+\mu+\kappa} \Gamma(\omega+\kappa) \Gamma(\omega+\mu) \Gamma(-\omega-\kappa-\mu)}{\Gamma(-\mu) \Gamma(-\kappa) \Gamma(2\omega+\mu+\kappa)}. \quad (10)$$

(d) *The case $\mu = 0$.* When $\mu = 0$ and $\omega + \kappa$ is an integer, (6) reduces to the result of Konetschny [1]

$$S = \frac{\pi^\omega (p^2 y)^{\omega+\kappa} (p \cdot n)^{2\nu+s} \Gamma(s+\nu+1/2) \Gamma(-\omega-\kappa-\nu) \Gamma(\omega+\kappa)}{\Gamma(-\kappa) \Gamma(-\nu) \Gamma(1/2+\omega+\kappa+\nu-s)}. \quad (11)$$

(e) *Light-cone gauge.* The light-cone gauge condition, $n^2 = 0^+$, corresponding to a special limit of $y \rightarrow +\infty$, is reached by analytic continuation via (7), yielding the result

$$\int d^2\omega q [(p-q)^2]^\kappa (q^2)^\mu (q \cdot n)^{\bar{\nu}} = \frac{\pi^\omega (p^2)^{\omega+\kappa+\mu} (p \cdot n)^{\bar{\nu}} \Gamma(\omega+\kappa) \Gamma(\omega+\mu+\bar{\nu}) \Gamma(-\omega-\kappa-\mu)}{\Gamma(-\kappa) \Gamma(-\mu) \Gamma(2\omega+\kappa+\mu+\bar{\nu})}. \quad (12)$$

As far as we know, a well-defined, Lorentz-invariant^{†1} regularization such as (12) for this gauge has not been obtained before. The striking similarity between (12) and (10) bodes well for the simplicity of this gauge. On the other hand, the gauge is also known to possess pathologies [13]. In particular the integral $S(p, n; -1, -1, -1, 1)$, but no others, regular in $n^2 \neq 0$ gauges, now has a (well-defined) double pole in ϵ . This gauge requires special scrutiny and will be discussed in detail elsewhere [14].

(f) *Poles.* As a generalization of primal integrals, the right-hand side of (6) has poles of $O(1/\epsilon)$ necessarily when at least one of $2 + K + M + N$, $-(2 + M + N + s)$ and $-(2 + K)$ is a non-negative integer. These three cases correspond respectively to the original Feynman integral being UV divergent, or IR divergent at either $q = 0$ or $q = p$. The representation is a meromorphic function of ν and is free from axial gauge singularities at $\nu = N$. It can be shown that, excepting the integral $S(p, n; -1, -1, -1, 1)$ mentioned earlier, the infinite and regular parts of all primal integrals are finite sums in powers of y with factors of $\ln y$. General results covering all primal integrals are given elsewhere [9].

(g) *Tadpole and volume integrals.* For any M and N , the tadpole integral

$$T(p, n; M, N) = \int d^2\omega q (q^2)^M (q \cdot n)^N, \quad (13)$$

^{†1} For a Lorentz-noninvariant regularization and other references to the light-cone gauge see ref. [13].

being either UV or IR divergent for all values of ω , cannot be regulated by dimensional regularization. However, treated as a subclass of S with $\kappa = 0$, it is regulated by (6) and is found to be zero, the value usually assigned to it without proof in dimensional regularization. In fact, all integrals in the new representation vanish whenever κ is a non-negative integer. It follows that the volume integral also vanishes: $\int d^2\omega q = 0$.

(h) *Two-loop integrals*. Because the new method is comparatively simple, analytical and removes the axial gauge singularity completely, we foresee no intrinsic difficulty in applying it to two- or higher-loop calculations which are thought to be prohibitively complicated with the PV prescription.

As a further demonstration of the power of (6) consider the subclass of integrals encountered in Yang–Mills theories and quantum gravity [1]

$$X(M, j) \equiv \int d^2\omega q (q^2)^M / [(p - q)^2 (q \cdot n)^j], \quad (14)$$

where M and $j \geq 1$ are integers. Express $j = 2N + s$ where $N \equiv [j/2]$ is the largest integer not larger than $j/2$ and $s \equiv \text{Mod}(j, 2)$. The integral may be classified in four categories:

(i) $M = i \geq 0, 3 + i - j \leq 0$. The integral is regular:

$$X(i, j) = X_{\text{reg}}(i, j) \equiv \frac{-\pi^2 y (p^2)^{1+i}}{(p \cdot n)^j (1/2 - N)(N + s - 1)} {}_3F_2 \left(\begin{matrix} 3 + i - j, -i, 1 \\ 3/2 - N, 2 - N - s \end{matrix} \middle| y \right), \quad (15)$$

where F^T is a truncated hypergeometric function

$${}_pF_q^T \left(\begin{matrix} a_p \\ b_q \end{matrix} \middle| y \right) \equiv \sum_{l=0}^L \frac{(a_p)_l y^l}{(b_q)_l l!}, \quad (16)$$

and L is the least of the non-negative integers in the set $\{0, -a_p, -b_q\}$.

(ii) $M = i \geq 0, 3 + i - j \geq 0$. The integral has an infinite part:

$$\begin{aligned} X(i, j) &= \theta(N + s - 2) X_{\text{reg}}(i, j) - \frac{\pi^2 (p^2)^{1+i} y^{N+s} \Gamma(1/2 - N) \Gamma(1 + i)}{(p \cdot n)^j \Gamma(3 + i - j) \Gamma(N + s)} \\ &\times \sum_{l=0}^{1+i-N-s} \frac{(-y)^l \Gamma(2 + i - N + l)}{\Gamma(1 + l) \Gamma(2 + i - N - s - l) \Gamma(1/2 + s + l)} [1/\epsilon + \ln(p^2 y \pi) - 2\psi(3 + i - j) \\ &+ 2\psi(i - N + 2 + l) - \psi(1 + l) + \psi(N + s + l) - \psi(1/2 + s + l)]. \end{aligned} \quad (17)$$

(iii) $M = -i < 0, 3 - i - j > 0$. The only possibility is $i = j = 1$, in which case it is the integral $S(p, n; -1, -1, -1, 1)$ mentioned earlier, and is equal to a regular infinite series representation. For $|y| \leq 1$,

$$X(-1, 1) = -\frac{\pi^{5/2} (p \cdot n)}{p^2 n^2} \sum_{l=0}^{\infty} \frac{\Gamma(1 + l)}{\Gamma(3/2 + l)} y^l [\psi(1 + l) - \psi(3/2 + l) + \ln y]. \quad (18)$$

(iv) $M = -i < 0, 3 - i - j \leq 0$. The integral has an infinite part:

$$\begin{aligned} X(-i, j) &= \theta(N + s - 2) X_{\text{reg}}(-i, j) + \frac{\pi^2 (p^2)^{1-i} y^{N+s} \Gamma(1/2 - N) \Gamma(i + j - 2)}{(p \cdot n)^j \Gamma(i) \Gamma(N + s)} \\ &\times \sum_{l=0}^{i+N-2} \frac{(-y)^l \Gamma(i + N + s - 1 + l)}{\Gamma(1 + l) \Gamma(1/2 + s + l) \Gamma(i + N - 1 - l)} [1/\epsilon + \ln(p^2 \pi / y) \\ &- 2\psi(i + j - 2) - 2\psi(i + N + s - 1 + l) + \psi(N + s + l) + \psi(1 + l) + \psi(1/2 + s + l)]. \end{aligned} \quad (19)$$

The $O(1/\epsilon)$ terms of some specific integrals belonging to types (ii) and (iv) have been previously derived by Capper and Leibbrandt [1] using the PV prescription and giving results equal to ours. The regular integral of (iii) has been reduced by van Neerven [15] and Konetschny [1] to a one-fold integral which has been shown [9] to be equivalent to the right-hand side of (18).

Efforts to find useful representations for three- and four-point and massive two-point functions are underway.

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