EXPONENT DERIVATIVES: AN ANALYTIC TECHNIQUE FOR REGULATING NONLINEAR FIELD EQUATIONS

H.C. LEE [1] and M.S. MILGRAM [2]
Atomic Energy of Canada Limited, Chalk River Nuclear Laboratories, Chalk River, Ontario, Canada K0J 1J0

Received 13 June 1983
Revised manuscript received 5 August 1983

An analytic technique for regulating the infinities of nonlinear equations of quantum gauge theories is described. The technique is ideally suited for dealing with high-order effects characterized by high powers of logarithms. The regulated equation, possessing high order poles with nonlogarithmic residues, is renormalizable.

Dimensional regularization [1] has been extremely useful for the renormalization program of gauge theories. The principal virtues of the method are: (i) it preserves gauge invariance and (ii) it analytically regulates the divergences in Feynman integrals so that they appear as poles in the ω-plane at the point $\omega = 2$, where 2ω is the generalized dimension of euclidean space–time. Although the renormalization program [2] for perturbation theories is by now well understood and conventional, the same is not true for nonperturbation theories. These theories are needed when perturbation does not work, either because the coupling is too strong, or the vacuum is nontrivial, or for some other reason. In lattice theories [3], the ultraviolet divergence is evaded by keeping the lattice spacing finite. However, the problem of the continuum limit, where the divergence re-emerges, is unresolved. A typical approach to a nontrivial vacuum [4] is the semiclassical one, where effects such as the vacuum polarization containing ultraviolet divergences are suppressed.

Another way of studying a nonperturbation theory is to seek solutions to nonlinear integral equations derived from the theory. An example of considerable current interest is the truncated Schwinger–Dyson equation [5] for the gluon propagator in quantum chromodynamics. Nonlinear equations, by themselves, are not peculiar to field theories since they are encountered in innumerable classical problems. The feature that sets the field-theoretical equation apart from classical ones is the presence of ultraviolet and other divergences of the integrals in the equation. If the divergent integrals could be easily regulated [6] (thereby allowing the equation to be renormalized), then the equation would be reduced to a classical form, to which a conventional method of analysis could be applied.

In this letter we introduce a method which should enable one to regulate analytically a class of field equations. The equations may include effects corresponding to any order of loop-expansion in perturbation theory. Furthermore, we demonstrate that the regulated equation is renormalizable.

The field theory we have in mind is the Yang–Mills sector of quantum chromodynamics with massless gluons. We work in an axial gauge [7,8], defined by the condition $A \cdot n = 0$, where A is the gauge field and n is an arbitrary auxiliary vector. Owing to the "spurious" singularity that appears in such a gauge, Feynman integrals in an axial gauge are notoriously difficult to evaluate [8]. Nevertheless, we will handle this task analytically. The motivation for working in an axial gauge is that the Faddeev–Popov ghosts [9] are decoupled so that the problem for the gluon may be reduced to solving a single equation, rather than a set of coupled ones.

[1] Theoretical Physics Branch.
[2] Applied Mathematics Branch.

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To demonstrate the proposal explicitly, consider the truncated Schwinger–Dyson equation [5] for the reduced gluon propagator – the two-point function Z with momentum p,

$$Z^{-1}(p,n) = 1 + g_0^2 \int d^4q\, K(q,p,n) Z(q,n)$$

$$+ \text{ integral with integrand quadratic in } Z\,. \qquad (1)$$

Here, n is the auxiliary vector for the axial gauge and K is a known kernel [5] which need not be specified for now. Other than the ultraviolet divergence, the integral in (1) may suffer from infrared divergence, resulting from the masslessness of the gluon, when either $q = 0$ or $p = q$, and from the spurious axial gauge singularity, when the contained factor $q \cdot n$ vanishes. If (1) were an ordinary integral equation, we might be able to solve it by expanding Z in independent functions (or powers) of the three scalars p^2, $p \cdot n$ and n^2 thereby reducing (1) to a set of algebraic equations for the coefficients in the expansion. The task at hand is more complicated than this for two reasons: (i) the integrals in (1) are divergent and must first be regulated, and (ii) the divergence of the integral implies that its regular part contains factors of $\ln p^2$, $\ln p \cdot n$ and $\ln n^2$. Since the equation is nonlinear, one must consequently expand Z in powers of these logarithms – Z is a "polylog" in the scalars. (The fact that Z^{-1}, but not Z, appears on the left-hand side of (1) is a source of complication. The expansion envisaged is at least viable for small p^2. Otherwise, techniques such as continuous fractions might have to be invoked.) Consequently integrals with integrands containing polylogs must be regulated and evaluated. That physical amplitudes contain polylogs is well known: divergent N-loop integrals generate logarithms of order N.

These considerations suggest that for a massless m-point function ($m \geqslant 2$), the generalized Feynman integral with external momenta p_i ($i = 1, \ldots, m-1$),

$$S_{2\omega}(\{p_i\}, n; \{\kappa_i\}, s) = \int d^{2\omega}q\, (q\cdot n)^{2\nu+s}(q^2)^{\mu}$$

$$\times \prod_{i=1}^{m-1} [(p_i - q)^2]^{\kappa_{i+2}} \qquad (2)$$

be studied, where κ_i (with $\kappa_1 \equiv \nu$ and $\kappa_2 \equiv \mu$) and ω are continuous variables and $s = 0$ or 1. The integral is free of all singularities if the (real parts of the) exponents lie within the region delineated by

$$\alpha_1 \equiv \omega + \sum_{i=1}^{m+1} \kappa_i < 0\,, \quad \nu + s > -\tfrac{1}{2}\,, \qquad (3a)$$

$$\omega + \mu + \nu + s > 0\,, \quad \omega + \kappa_i < 0 \quad (i = 3, \ldots, m+1)\,. \qquad (3b)$$

This region exists, at least in the vicinity of $\omega = 2$, $\kappa_i = -1$, $s = 1$. To regulate S, it is therefore sufficient to find a representation of S in the region defined by (3); for variables exterior to the region the integral is well-defined by analytic continuation [6]. Integrals with logarithmic integrands are evaluated using "exponent derivatives" defined as:

$$S_{2\omega}^{\{j_i\}}(\ldots) \equiv \prod_{i=1}^{m+1} (d/d\kappa_i)^{j_i} S_{2\omega}(\ldots)$$

$$= \int d^{2\omega}q\, (q\cdot n)^{2\nu+s}(q^2)^{\mu} \ln^{j_1}(q\cdot n)^2 \ln^{j_2}(q^2)$$

$$\times \prod_{i=1}^{m-1} \{[(p_i - q)^2]^{\kappa_{i+2}} \ln^{j_{i+2}}(p_i - q)^2\}\,, \qquad (4)$$

plus a prescription to be given shortly.

To evaluate (2), each factor in the integrand is replaced by Euler's representation for the gamma function

$$a^{\kappa} = \frac{1}{\Gamma(-\kappa)} \int_0^{\infty} t^{-\mu-1} e^{-at}\, dt\,, \qquad (5)$$

after which the q-integral assumes a known form allowing the integration to be carried out [8]. The next step is to apply the transformation

$$\kappa_1 = \lambda\tau_1\,, \quad \kappa_i = \lambda\tau_i \prod_{l=1}^{i-1} (1 - \tau_l) \quad (i = 2, \ldots, m)\,,$$

$$\kappa_{m+1} = \lambda \prod_{l=1}^{m} (1 - \tau_l)\,, \qquad (6)$$

with λ varying from 0 to ∞, and each of the τ_i's varying from 0 to 1. The jacobian determinant is $\lambda^m \Pi_{i=1}^m (1 - \tau_i)^{m-i}$. Of the $m+1$ parametric integrations, two can be immediately evaluated: the λ(scale)-integration is trivially performed using (5), thereby isolating the factor $\Gamma(-\alpha_1)$ containing the pole for the ultraviolet divergence. The τ_1-integration can also be carried out, regulalating the axial gauge singularity. What remains is a nontrivial, $(m-1)$-fold integral in which reside the infrared singularities corresponding to (3b).

For $m = 2$, the case relevant to the two-point function of (1), a representation is $[\kappa \equiv \kappa_3$ in (2)]:

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s)$$

$$= \frac{\pi^{\omega}(p^2)^{\alpha_1}(n^2)^{\alpha_2}(p\cdot n)^s\Gamma(\alpha_2+s+1/2)}{\Gamma(\beta_1-\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(-\alpha_0-\alpha_1-s)\Gamma(-\alpha_2)}$$

$$\times G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1+\alpha_0, & 1+\alpha_1, & 1+\alpha_2\\ 0; & \beta; & 1/2-s\end{matrix}\right), \qquad (7)$$

where $y = (p\cdot n)^2/(p^2n^2)$, $\alpha_0 = -(\omega+\mu+\nu+s)$, $\alpha_1 = \omega+\kappa+\mu+\nu$, $\alpha_2 = \nu$, $\beta_1 = \omega+\kappa+\nu$ and the symbol G stands for a Meijer G-function [10], a well-defined function of all its variables and parameters.

The regularization method used here is different from the standard dimensional regularization [1] in two important aspects: (a) In addition to the dimension of space–time (2ω), the exponents κ_i are generalized to continuous variables; this is mandatory in order that (2) be meaningful. (b) The axial gauge singularity, originating from the factor $(q\cdot n)^{2\nu+s}$ is regulated analytically rather than by the widely used principal-value prescription [11]. The power and many advantages of this new, hybrid dimensional and analytic [12] method over the standard one have been detailed elsewhere [13].

The right-hand side of (7) has poles when the indices α_1, α_0 or $\alpha_2-\beta_1$ are nonnegative integers, corresponding respectively to the ultraviolet and the infrared divergences – at $q^2 = 0$ and $(p-q)^2 = 0$ – of the original integral. Significantly, it does not suffer from axial gauge singularities. If the limits $\kappa, \mu, \nu \to$ integers are taken before $\epsilon \equiv \omega^{-2}$ is allowed to approach zero, then all poles are of $O(1/\epsilon)$. It has been shown [13] that the standard method [11] gives results identical to (7), for both the infinite and the regular part of the integral. The properties of representation (7) have been discussed in detail in ref. [13].

We now discuss the renormalizability of (1), when (7) and its exponent derivatives as defined by (4) are used to evaluate integrals generated by the expansion of Z in polylogs. As the criterion for renormalizability we demand that all infinite parts be a series of multiple (or single) poles in ϵ, with residues free of logarithms. This criterion ensures that the infinite parts can be absorbed into renormalization constants and then be cancelled by local counterterms [2]. It is easily seen that the poles of (7) do not have logarithmic residues, a characteristic of all one-loop integrals. Before giving a prescription ensuring that exponent derivatives, to all orders, do not generate logarithmic infinite parts we first give a heuristic argument indicating such a result is to be expected. In perturbation theories, logarithmic infinite parts are generated by overlapping divergencies in multi-loop integrals. They are eliminated, according to a prescription of 't Hooft and Veltman [1], by subtracting $(N-1)$-loop integrals inserted with appropriate counterterms from N-loop integrals. For example, a two-loop integral with its subtraction appears schematically as

$$\int d^{2\omega}q \int d^{2\omega}q'(\ldots) - \frac{1}{\epsilon}\int d^{2\omega}q\,(\ldots)$$

$$= \int d^{2\omega}q\,(\ldots)[(q^2)^{\epsilon}/\epsilon - 1/\epsilon]\,, \qquad (8)$$

where divergent integrands are implicit. In the limit $\epsilon \to 0$ the square bracket is similar to an exponent derivative: $\lim_{\kappa\to 0}\,[d(q^2)^{\kappa}/d\kappa] = \ln q^2$. Although the right-hand side of (8) is not really a derivative because the limit $\epsilon \to 0$ can be taken only after the integration, the similarity between (8) and (4) is nevertheless suggestive.

Because of the inherent ambiguities of generalizing a function from a set of integers, the definition of the exponent derivatives in (4) is incomplete unless a limiting process is prescribed. Our prescription is based on the observation that the poles associated with the indices α_0, α_1 and $\alpha_2-\beta_1$ are of order $1/\epsilon_0$, $-1/\epsilon_1$ and $1/\epsilon_3$, respectively in three independent ϵ variables. In the limit $(\kappa, \mu, \nu) \to$ integer they become equal (to $1/\epsilon$); this can be expressed as

$$1/\epsilon_i = -1/\epsilon_1 + \sigma_i/\epsilon_1\epsilon_i\,, \quad i = 0, 3\,. \qquad (9)$$

The variable $\sigma_i \equiv \epsilon_1 + \epsilon_i$ depends on (κ, μ, ν) but not on ω, and vanishes in the limit $(\kappa, \mu, \nu) \to$ integers. In our prescription, the $O(\sigma_i)$ term in (9) is discarded. This term has no bearing on primal integrals (since it is equal to zero) but effects exponent derivatives and is related to the infrared divergence. Now the exponent derivatives are uniquely defined [13]. They are independent of the order of differentiation and generate renormalizable high-order poles and their associated logarithms only via the identity

$$(\mathrm{d}/\mathrm{d}\epsilon)^N [(p^2)^\epsilon/\epsilon] = (-)^N N!/\epsilon^{N+1}$$

$$+ [\ln^{N+1}(p^2)]/(N+1) + \mathrm{O}(\epsilon)\,, \qquad (10)$$

in which poles in ϵ with logarithmic residues are manifestly absent.

To illustrate the technique consider the integral

$$I_1 = \int \mathrm{d}^{2\omega} q \, \frac{(p\cdot q)[p\cdot(p-q)][q\cdot(p-q)]}{(p-q)^2 q^2 (q\cdot n)[(p-q)\cdot n]} \qquad (11)$$

encountered in the solution of (2) in the infrared limit and at the "one-loop" level. The integral is simplified by evaluating partial fractions

$$1/(q\cdot n)[(p-q)\cdot n] = (1/p\cdot n)[1/q\cdot n + 1/(p-q)\cdot n]$$

and changing variable

$$q \to p - q$$

when necessary. The numerator in (11) can be re-expressed as

$$(p\cdot q)[p\cdot(p-q)][q\cdot(p-q)]$$

$$= \tfrac{1}{8}(p^6 - p^4 q^2 - p^2 q^4 + q^6) + \mathrm{O}((p-q)^2)\,. \qquad (12)$$

Terms with at least one positive power of $(p-q)^2$ may be ignored because all primal integrals with $\kappa \geqslant 0$ are identically zero [13]. The integral I_1 can now be expressed as a sum of nonzero S-integrals:

$$I_1 = [1/4(p\cdot n)]\,[p^6 S(-1,-1,-1,+1)$$

$$- p^4 S(-1,0,-1,1) - p^2 S(-1,1,-1,1)$$

$$- S(-1,2,-1,1)]\,, \qquad (13)$$

where $S(\kappa,\mu,\nu,s) \equiv S_{2\omega}(p,n;\kappa,\mu,\nu,s)$. These integrals are found [14] to be

$$p^6 S(-1,-1,-1,1) = S_0 Z_1/2\,, \qquad (14a)$$

$$p^4 S(-1,0,-1,1) = S_0 a_0\,, \qquad (14b)$$

$$p^2 S(-1,1,-1,1) = S_0(1-2y)a_0\,, \qquad (14c)$$

$$S(-1,2,-1,1) = S_0\{[1-(16/3)y(1-y)]a_0$$

$$+ (8/9)y(1-y)\}\,, \qquad (14d)$$

where $S_0 \equiv -2\pi^\omega p^4 (p\cdot n)/n^2$, $a_0 \equiv 1/\epsilon + \ln(4p^2 y) + \gamma - 2$, $\epsilon = \omega - 2$, $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant and

$$Z_m = \sqrt{\pi}\sum_{l=0}^{\infty} \frac{y^l \Gamma(m+l)}{\Gamma(m+1/2+l)}$$

$$\times [\psi(m+l) - \psi(m+1/2+l) + \ln y]\,. \qquad (15)$$

Substitution of (14) into (13) yields

$$I_1 = (\pi^\omega p^4/2n^2)[(1 + \tfrac{10}{3}y - \tfrac{16}{3}y^2)a_0$$

$$- \tfrac{8}{9}y(1-y) - \tfrac{1}{2}Z_1]\,. \qquad (16)$$

The integral happens to have a finite $y \to 0$ limit (corresponding to the special gauge [15] $p\cdot n = 0$, p^2 and n^2 finite):

$$\lim_{y\to 0} I_1 = (\pi^2 p^4/2n^2)[1/\epsilon + \ln p^2\pi + \gamma$$

$$+ \mathrm{O}(y\ln y, y)] \qquad (17)$$

which is, however, not a general property of integrals obtained from (2).

Another integral [15] that happens to be ultraviolet- and infrared-finite but is singular at $y = 0$ is [16]

$$I_2 = \int \mathrm{d}^{2\omega} q \, \frac{p^2 q^2 - (p\cdot q)^2}{(p-q)^2 q^2 (q\cdot n)^2 [(p-q)\cdot n]^2}$$

$$= -\tfrac{1}{2}(p\cdot n)^{-2}[p^4 S(-1,-1,-1,0)$$

$$- 2p^2 S(-1,0,-1,0) + S(-1,1,-1,0)]$$

$$- (p\cdot n)^{-3}[p^4 S(-1,-1,-1,1)$$

$$- 2p^2 S(-1,0,-1,1) + S(-1,1,-1,1)]$$

$$= (\pi^2/n^4)(4 + Z_2)\,, \qquad (18)$$

where

$$\lim_{y\to 0} I_2 = (4\pi^2/3n^4)(-\tfrac{13}{6} + \ln 4y) + \mathrm{O}(y\ln y, y)\,. \qquad (19)$$

An example of integrals with higher powers of $1/q\cdot n$ is

$$\int \mathrm{d}^{2\omega} q\,[q^2(p-q)^2(q\cdot n)^4]^{-1} = (8\pi^2/3p^4 n^4) \qquad (20)$$

$$\times \{(1-4y)[1/\epsilon + \ln(p^2\pi/4y) + \gamma] - 1/4y - 4 + 6y\}$$

As examples of exponent derivatives we consider those of $S(-1,0,-1,1)$. From (4), (9) and (10), the results are

$$\frac{\mathrm{d}}{\mathrm{d}\kappa} S(-1,0,-1,1) = S_1 \left[a_1 + \tfrac{1}{12}\pi^2 + 2\ln(4yp^2) - 4\right] , \tag{21a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\mu} S(-1,0,-1,1) = S_1 \left[a_1 - a_0 - \tfrac{1}{12}\pi^2 + 2\ln p^2 + \ln 4y + \tfrac{1}{2}\ln^2 4y + \tfrac{1}{2} y\bar{Z}\right] , \tag{21b}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\nu} - \frac{\mathrm{d}}{\mathrm{d}\mu} - \ln n^2\right) S(-1,0,-1,1) = \int \mathrm{d}^{2\omega} q \, \frac{\ln[(q\cdot n)^2/q^2 n^2]}{(p-q)^2 (q\cdot n)} = S_1 \left[(1+\ln 4) a_0 + \tfrac{1}{3}\pi^2 - 4 + 2\ln 4y - \tfrac{1}{2}\ln^2 4y - \tfrac{1}{2} y\bar{Z}\right] , \tag{21c}$$

where $S_1 = 2\pi^\omega (p\cdot n)/n^2$, $a_1 = 1/\epsilon^2 + \gamma/\epsilon + \gamma^2/2 - \frac{1}{2}\ln^2(4yp^2)$ and

$$\bar{Z} = \sqrt{\pi} \sum_{l=0}^{\infty} \frac{y^l \Gamma(3+l)\Gamma(1+l)}{\Gamma(2+l)\Gamma(5/2+l)} \left[\psi(3+l) + \psi(1+l) - \psi(2+l) - \psi(5/2+l) + \ln y\right] . \tag{21d}$$

A program has been written [14], using SCHOONSCHIP [17], to evaluate all primal integrals (7) and their exponent derivatives of any order [14]. Work on solving the Schwinger–Dyson equation using techniques described here is under way.

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