THE BAG MODEL AND THE NAMBU–GOLDSTONE PION

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The MIT bag model for the pion is improved and extended in such a way that the pion does not have spurious center-of-mass motions; perturbative gluon contributions to the pion mass m_π and decay constant f_π are both calculated to lowest order in α_s. The pion is a Nambu - Goldstone boson in the sense that the vacuum in the bag refers to massive constituent quarks, but not so massless current quarks. The transformation of Nambu and Jona-Lasinio between massive and massless quarks is utilized in the computation of f_π, the result of which strongly suggests that quarks in the pion are correlated, characterized by a correlation momentum which is ~ 300 MeV/c. The vacuum expectation value for the massless quark condensate is calculated to be $\langle\bar{\psi}\psi\rangle \sim 0.04\ \mathrm{GeV}^3$, corresponding to a current quark mass of ~4 MeV. The requirement that m_π approaches zero in a manner consistent with PCAC constrains the bag energy to be $m_\pi/4$.

Because of its spin-parity (0^-) and its having a mass very much smaller than that of any other hadron, the pion has long been considered as the logical candidate for the Nambu–Goldstone boson associated with the spontaneous breaking of the chiral symmetry of the strong interaction [1,2]. In this scheme it is assumed that chiral symmetry is broken by the quark mass term $m_0\bar{\psi}\psi$. In the limit $m_0 \to 0$ but with the vacuum expectation value $\langle\bar{\psi}\psi\rangle \neq 0$, chiral symmetry is spontaneously broken and associated with it is a massless Nambu–Goldstone boson (i.e. a massless pion). The physical nearly massless pion corresponds to the Nambu–Goldstone boson when $m_0 \neq 0$ and $\langle\bar{\psi}\psi\rangle \neq 0$. A very important consequence of the hypothesis described above is PCAC (partially conserved axial vector current) [3,2], the use of which permeates our understanding of nuclear physics. A key element of PCAC is that the pion decay constant remains finite in the limit of vanishing pion mass.

In this work we appraoch the subject from the point of view of the MIT bag model [4] in which the problem of quark confinement is not solved but is simulated by a cavity subjected to an inward pressure B and in which the pion is considered ot be a quark–antiquark composite bound within the pressurized cavity. Because of its amenability to calculation the bag model has become a powerful tool for studying hadrons at low energies [4]. Our prime objective is to examine to what extent the bag model pion can be reconciled with the idea that it is a Nambu–Goldstone boson. This work was inspired by the earlier work of Donoghue and Johnson (DJ) [5], but our approach is drastically different from theirs and we have gone considerably farther than DJ toward our objective. Our model for the pion differs from the MIT model in three important aspects:
(i) the pion state does not have spurious center-of-

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mass motion [5,6], (ii) f_π is treated on the same footing as m_π both are calculated in perturbation theory to order α_s, the effective coupling constant of QCD; this allows one to retain the desirable PCAC property that f_π remains finite in the limit $m_\pi \to 0$. (iii) we distinguish the massive constituent quarks of which the pion is composed from the massless (or almost massless) current quarks of the bare, (almost) chirally symmetric lagrangian, and recognize the notion that the vacuum in the bag refers to constituent quarks, but not to current quarks; utilizing a transformation similar to that of Nambu and Jona–Lasinio (NJL) [7] between the constituent and current quark operators, we establish a connection between f_π and the vacuum expectation value $\langle\bar{\psi}\psi\rangle$.

In this paper we give the important results of our work; details will be given elsewhere [8]. Here we merely mention that, compared to previous work [4, 5], we have also considerably simplified the computation by working in the momentum representation. We find that the gluon contribution to f_π is important and that f_π is very sensitive to quark correlation in the pion wavefunction which suppresses the high momentum spectrum, and which can be characterized by a correlation momentum p_c. In the limit $p_c \to \infty$, there is no correlation and we find $f_\pi \sim 300$ GeV and $\langle\bar{\psi}\psi\rangle \sim 0.1\ \mathrm{GeV}^3$. When $p_c \sim 300$ MeV, f_π has the correct value ($\sim$130 MeV) and $\langle\bar{\psi}\psi\rangle \sim 0.04\ \mathrm{GeV}^3$.

We write the pion with momentum $\boldsymbol{P}$ as a quark–antiquark composite

$$|\pi(\boldsymbol{P})\rangle = 2E_{\mathrm{p}}\int \mathrm{d}^3p_1\, \mathrm{d}^3p_2\, \delta(\boldsymbol{P}-\boldsymbol{p}_1-\boldsymbol{p}_2)F(P,p) \times \sum_{s_1 s_2} \bar{\mathrm{u}}(p_1 s_1)\gamma_5 \mathrm{v}(p_2 s_2) b^+_{p_1 s_1} d^+_{p_2 s_2}|0\rangle, \qquad (1)$$

with the normalization

$$\langle\pi(\boldsymbol{P})|\pi(\boldsymbol{P}')\rangle \equiv \delta(\boldsymbol{P}-\boldsymbol{P}')2E_{\mathrm{P}}N^2(P), \qquad (2)$$

where b_i^+ (d_i^+) creates a quark (antiquark) of momentum p_i, mass m_i and polarization s_i; $p = \frac{1}{2}(p_1 - p_2)$ is the relative momentum; $F(P,p) = f(p_1)f(p_2)\xi(p)$ is the quark–antiquark wavefunction. For the MIT bag

$$f(p_i) = f(x_i, z_i)$$

$$\equiv R^3[j_0(x_i - z_i) - j_0(x_i + z_i)]/2x_i z_i,$$

where R is the bag radius, $z_i = p_i R$, and x_i is the eigenmode solution [4] of the bag: $x_i - [1 - \mu_i - (x_i^2 + \mu_i^2)^{1/2}]\tan x_i = 0$, $\mu_i = m_i R$. For the correlation function we choose $\xi(p) = [1 + (p/p_c)^2]^{-1}$, which has the effect of suppressing the wavefunction for momentum greater than the correlation momentum p_c. We do not derive $\xi(p)$, rather we assume it to be a consequence of QCD, and shall show that the bag model of the pion is much improved by its presence; for the results reported here the detail functional form of $\xi(p)$ is not important. When integrated over all $\boldsymbol{P}$, the wavefunction in (1) becomes identical to that of the MIT bag model in the limit $p_c \to \infty$. In our model the pion at rest has $\boldsymbol{P} = 0$. For simplicity we assume the quark and antiquark in the pion have the same mass, $\mu_1 = \mu_2 = \mu$.

The effective two-body quark–quark interaction is derived from the color–electromagnetic energy of the gluon fields:

$$H_{\mathrm{g}} = -2\pi\alpha_s \int \mathrm{d}^3x\, \partial_\nu A^a_\mu \partial^\nu A^{a\mu}, \qquad (3)$$

$$A^a_\mu(x) = \int \mathrm{d}^4y\, D_{\mathrm{F}}(x-y)\bar{\psi}(y)\tfrac{1}{2}\lambda^a\gamma_\mu\psi(y), \qquad (4)$$

where $\psi(y)$ is the quark field, λ^a ($a = 1, ..., 8$) are generators of the SU(3) color group, and we use the static approximation for the gluon propagator

$$D_{\mathrm{F}}(x-y) = \delta(x_0 - y_0)(4\pi|\boldsymbol{x}-\boldsymbol{y}|)^{-1}. \qquad (5)$$

To order α_s the mass of the pion can be written as

$$E_\pi = E_0 + E_{\mathrm{g}} = E_{\mathrm{K}} - E_{\mathrm{Z}} + E_{\mathrm{B}} + E_{\mathrm{g}}, \qquad (6)$$

where $E_{\mathrm{Z}} = Z/R$ is the zero point energy, $E_{\mathrm{B}} = (4\pi/3) \times BR^3$ is the bag energy, and the kinetic energy E_{K} is calculated from the free hamiltonian $H_{\mathrm{F}} = \Sigma_{i=1}^2 \gamma^0(\boldsymbol{\gamma}\cdot\boldsymbol{p}_i + m_i)$,

$$E_{\mathrm{K}} = \langle\pi(0)|H_{\mathrm{F}}|\pi(0)\rangle/\langle\pi(0)|\pi(0)\rangle = I_3/RI_2 \equiv 2x/R,$$

$$I_n = \int_0^\infty \mathrm{d}z\, z^2\phi^2(z)[2\omega]^n, \quad \phi(z) = f^2(x,z)\xi(z), \qquad (7)$$

and $\omega = \omega(z) = (z^2 + \mu^2)^{1/2}$. Z and B are parameters of the model. The gluon energy is

$$E_{\mathrm{g}} = \langle\pi(0)|H_{\mathrm{g}}|\pi(0)\rangle/\langle\pi(0)|\pi(0)\rangle = -(2\alpha_s/3\pi)I_{\mathrm{g}}/RI_2,$$

$$I_{\mathrm{g}} = 16\int_0^\infty \mathrm{d}z\, \mathrm{d}z'\, \phi(z)\phi(z')zz'\omega\omega' \ln\left|\frac{z+z'}{z-z'}\right|. \qquad (8)$$

To first order in α_s the pion decay constant is given by

$$f_\pi = f(0) + f_g(0), \tag{9}$$

where

$$f(P)P_\mu = \langle 0|J_{\mu 5}|\pi(P)\rangle / N(P),$$

$$f_g(P)P_\mu = \langle 0|J_{\mu 5}(E_0 - H_0)^{-1} H_g|\pi(P)\rangle / N(P), \tag{10}$$

H_0 is the hamiltonian corresponding to E_0 and $J_{\mu 5} = \bar{\psi}\gamma_\mu\gamma_5\psi$ is the axial vector current in terms of current quark fields. The axial vector current can be re-expressed in terms of fields of constituent quarks with the aid of the momentum dependent transformation [7] between constituent and current quarks ‡1. This yields an extra factor $\cos^2\theta_p - \sin^2\theta_p = \beta_p \equiv p^2/(p^2 + m^2)$ to the operator; $\theta_p = \tan^{-1}[(1 - \beta_p)/(1 + \beta_p)]^{1/2}$ is the angle of transformation where the small mass m_0 of the current quarks is ignored. We then have

$$f(0) = (\Omega/R)I_d, \quad \Omega = (16\pi E_0 R/I_2)^{1/2},$$

$$I_d = \mu \int_0^\infty dz\, z^3 \phi(z)\xi(z)/\omega^2, \tag{11}$$

$$f_g(0) = -(2\alpha_s/3\pi)(\Omega/R)I_{dg},$$

$$I_{dg} = 2\mu \int_0^\infty dz\, \phi(z)\, \mathcal{P}\int_0^\infty dz' \frac{\omega z z'^2}{\omega'^3(\omega' - x)} \times \eta(z')\ln\left|\frac{z + z'}{z - z'}\right|, \tag{12}$$

$\mathcal{P}$ is the principal value operator and x is defined in (7). Since $E_\pi = E_0 + E_g$, formally f_π need not vanish with E_π, but is proportional to $(E_0)^{1/2} = (-E_g)^{1/2}$ when $E_\pi \to 0$.

In (11) and (12) we have assigned a cutoff function η to the vacuum. This is necessary when we realize that without a cutoff, the vacuum expectation value for the current quark density, or the quark condensate, is logarithmically divergent. Summing over two flavors, three colors and the positive and negative energy states, the current quark condensate is, to O(1)

$$\langle\bar{\psi}_0\psi_0\rangle = \frac{12}{(2\pi)^3}\int d^3p \frac{m}{(p^2 + m^2)^{1/2}} \sin^2\theta_p\, \eta(p)$$

$$= (3m^3/\pi^2)[\ln(2\Lambda_c/m) - 1 + O(m/\Lambda_c)], \tag{13}$$

where Λ_c is the cutoff momentum. To this order, an identical expression is obtained when we let η have the same functional form as the correlation function ξ. Notice that to within a logarithm, the condensate vanishes as the third power of the constituent quark mass when the latter approaches zero.

For the parameters of our model we use the value $B^{1/4} = 131$ MeV, for reasons to be made clear later, and the QCD inspired relation [5] $\alpha_s(R) = \alpha_0/\ln(1 + 1/R\Lambda)$ for the R dependence of the running coupling constant, where empirically [10] $\Lambda = 0.1$–0.5 GeV. We use $\Lambda = 0.2$ GeV, and $\alpha_0 = 0.5$; at the pion radius ($R \sim 0.6$ fm) this gives $\alpha_s \sim 0.5$ and about the correct amount of gluon energy ($E_g \sim -0.6$ GeV) needed for the splitting of the π and ρ masses. Taking now the energy as a function of the two parameters m and Z and the variable R, we can solve for m and Z by requiring that $E_\pi = m_\pi$ be stationary, or $dE_\pi/dR = 0$, at $R = 0.6$ fm, the experimental value for the pion radius. This still leaves the correlation momentum p_c undetermined. The results in table 1 illustrate the dependence of the various calculated quantities on p_c; there is no correlation when $p_c = \infty$. Values for the decay constant given in the table corresponds to $\Lambda_c = \infty$, but those values are not sensitive to Λ_c, provided it is $\gtrsim 7$ GeV, which is the value for Λ_c used to compute the quark condensate. The calculated results are in best agreement with the empirical value $m \sim 340$ MeV, $E_g \sim 600$ MeV and $f_\pi = 132$ MeV when $p_c = 300$ MeV. When p_c is greater than 300 MeV the calculated values of the above quantities rapidly become too large. This implies that the kinetic motion ascribed to the quark with the wavefunction determined from boundary conditions of the bag model alone is too hard. Here we have softened this motion with a correlation function.

One of the features of our model is that the pion decay constant, like the pion mass, is also the result of cancelling contributions. This is to be expected since the typical energy scale of the bag model is $\gtrsim 0.5$ GeV, and in this model it would be difficult to generate energies such as m_π and f_π, which are of the

‡1 The exact form of the correct transformation between constituent and current quarks is not known. See, however, ref. [9]. The NJL transformation used here ignores the spin but satisfies the most important requirement of the correct transformation: $\theta_p \to 0$ in the limit of $m/p \to 0$.

Table 1
Results a) for pion calculated with several different values for p_c.

	p_c			
	150	300	500	∞
Z	0.842	1.26	1.34	2.37
m	283	352	386	518
R_π (fm) b)	0.6	0.6	0.6	0.6
α_s c)	0.514	0.514	0.514	0.514
E_K	925	1157	1323	1787
E_g	-543	-639	-777	-903
E_B d)	35	35	35	35
E_Z	-277	-413	-442	-780
$f(0)$	638	748	873	1050
$f_g(0)$	-554	-622	-636	-778
$f_\pi = f(0) + f_g(0)$	84	126	237	272
$\langle\bar{\psi}\psi\rangle$ (10^{-2} GeV^3)	2.0	3.6	4.5	9.7

a) Unless otherwise stated, all energies (momenta) are in units of MeV (MeV/c).
b) Input. c) $\alpha_s = 0.5/\ln(1 + 1/R\Lambda)$, $\Lambda = 200$ MeV.
d) The bag constant is constrained such that $E_B = m_\pi/4$; at $R = 0.6$ fm, $B^{1/4} = 131$ MeV; see text for details.

order of 0.1 GeV, without cancellation effects.

When $p_c = 150$–300 MeV, the quark condensate is calculated to be $(2–4) \times 10^{-2}$ GeV^3. This quantity is not directly measurable, but recently reported Monte Carlo calculations [11] in lattice QCD also yielded $\langle\bar{\psi}\psi\rangle \sim 3 \times 10^{-2}$ GeV^3. If we use the PCAC sum rule [2,12],

$$m_0\langle\bar{\psi}\psi\rangle = \tfrac{1}{2}f_\pi^2 m_\pi^2, \tag{14}$$

then we deduce $m_0 \sim 7$ MeV for the current quark mass, a value which is consistent with other phenomenological estimates. Also notable is that our results satisfy approximately $f_\pi \propto \langle\bar{\psi}\psi\rangle^{1/2}$, which in turn implies that $m_\pi \propto m_0^{1/2}$.

On the other hand, since m_0 does not appear explicitly in our calculation, there seems to be no obvious mechanism within our model to dynamically drive the pion mass to zero. What we know is that, assuming the zero mass pion to be a Nambu–Goldstone boson, whatever mechanism we choose must be such that m, R, α_s and p_c remain essentially unchanged. The only parameters that can be changed are therefore B and Z. It can be shown that if $E_\pi(B,Z) = m_\pi$ is a stationary solution, then so is $E_\pi(B_0,Z_0) = 0$, provided that $B_0 = B - (3/16\pi)m_\pi R^{-3}$ and $Z_0 = Z + 3m_\pi R/4$. Since the bag pressure violates chiral invariance, the bag constant B_0 should vanish in the symmetry limit, i.e. when $m_\pi = 0$. This suggests that in the bag model the bag pressure plays the role of m_0 for explicit chiral invariance violation; it follows that $(16\pi/3)BR^3 = m_\pi$, or the bag energy be equal to one-fourth the pion mass. At $R = 0.6$ fm, this yields $B^{1/4} = 131$ MeV, thus explaining our reason for choosing to use this value. It is remarkable that this value is so close to the ~140 MeV determined phenomenologically for the MIT bag [4,5]. A plausible interpretation for the increase of Z in the limit $m_\pi \to 0$ is that E_Z represents the energy of the zero point motion of the current quarks, which should increase when these quarks become massless.

The procedure described above clearly decouples f_π and $\langle\bar{\psi}\psi\rangle$ from m_π as $m_\pi \to 0$, thereby allowing, through (14), the Nambu–Goldstone–PCAC relations $\lim_{m_0\to 0} m_\pi \propto m_0^{1/2}$ and $\lim_{m_\pi\to 0} f_\pi \propto \langle\bar{\psi}\psi\rangle^{1/2}$ = constant $\neq 0$ to be satisfied.

We conclude that the bag model and the Nambu–Goldstone descriptions of the pion can be made compatible, provided a clear distinction is made between massive constituent quarks, in terms of which the structure of the pion is simple, and the (almost) massless current quarks which define the QCD lagrangian. The bag and its associated vacuum refer to the former, but not to the latter. For the bag to produce numerical results that agree with experimental data, it is necessary to suppress components of the quark wavefunction of the MIT bag with momentum greater than 300 MeV. These conclusions are very similar to those from the recent work of Goldman and Haymaker [13] [‡2], whose approach of the topic under discussion is quite different from ours. We calculated the (current) quark condensate to be $\langle\bar{\psi}\psi\rangle = \sim 4 \times 10^{-2}$ GeV^3, which corresponds to a current quark mass of ~4 MeV. We argue that in the bag model the only way that is consistent with PCAC to drive the pion mass to zero is to vary the parameters B and Z in a prescribed fashion, such that $B \to 0$ when $m_\pi \to 0$; this constrains the bag energy to be one quarter of the pion mass.

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