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TWO YANG-MILLS THEORIES IN THE LIGHT-CONE GAUGE: Complete one-loop counterterms

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The Mandelstam-Leibbrandt prescription is used to study the one-loop structures of the two-component (LCM2) and four-component (LCM4) formalisms of the same Yang-Mills theory in the light-cone gauge. The complete one-loop counter lagrangians are constructed by computing the one-loop two-, three- and four-point vertices. LCM2 is renormalizable order-by-order in g with $\delta\mathcal{L}_{\text{counter}} = (Z-1)\mathcal{L}$, $Z = 1 + 11g^2C_2/48\pi^2\varepsilon$. For LCM4, both the two- and three-vertices generate anomalous counterterms which, however, cancel upon summation so that the total $\delta\mathcal{L}_{\text{counter}}$ is the same as LCM2. Slavnov-Taylor identities are satisfied in LCM4; they do not exist in LCM2. The method of analytic regularization is used in computation; all invariant and tensor integrals are evaluated using a single representation for light-cone invariant two-point integrals. The calculation is exceedingly simple in LCM2, far less so in LCM4.

1. Introduction

The light-cone gauge has long been recognized as potentially a most simple and useful gauge [1] for the study of nonabelian gauge theories. It has recently become popular in the study of supersymmetric theories [2]. The gauge is a special axial ghost-free gauge defined by the constraint

$$\underline{A}\cdot n_+ \equiv A^+ = 0, \tag{1.1}$$

where $\underline{A}$ is the Yang-Mills field and n_+ is one of the two independent, null, light-cone vectors in Minkowski space (for notation see sect. 2).

In addition to being ghost-free, the light-cone gauge has special simplifying features arising from the properties of the null vectors n_+, and from the elimination of the second light-cone component of the Yang-Mills field, $\underline{A}^- \equiv \underline{A}\cdot n_-$, via the Euler lagrangian equation,

$$\partial^+\underline{A}^- = \partial^i\underline{A}^i + g(\partial^+)^{-1}(\underline{A}^i\times\partial^+\underline{A}^i), \tag{1.2}$$

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thereby reducing the effective theory to one involving a two (instead of four)-component Yang-Mills field. The light-cone gauge with the unphysical $\underline{A}_{\pm}$ components thus eliminated is sometimes called the physical gauge.

On the other hand, the peculiarity of the singular properties of the light-cone gauge has prevented it from becoming more widely used. The difficulty in working with the light-cone gauge can be simply put: a practical prescription for regulating divergent Feynman integrals was lacking.

The principal-value prescription, which has been successfully used in dealing with the singularities of the closely related axial gauge ($\underline{A}\cdot n=0$, $n^2\neq 0$) [3], has recently been firmly established as being inadequate for the light-cone gauge [4–7]; the prescription disobeys the rule of power counting, rendering the gauge unrenormalizable at the one-loop level. This defect has been brought into even sharper focus by the recent demonstration by Mandelstam [8] and by Brink et al. [9] that the $N=4$ supersymmetric Yang-Mills theory would be ultraviolet-finite to all orders in the light-cone gauge, and therefore finite in any other gauge, if integrals in the light-cone gauge obeyed power counting.

Happily, Mandelstam [8] also proposed a prescription which possesses the analytic property necessary for preserving power counting. Another prescription that appears to have the same attribute was later proposed by Leibbrandt [6]. These two prescriptions have recently been used by several groups to study the self-energies of simple and supersymmetric Yang-Mills theories in the light-cone gauge at the one-loop level [5,10,11]. A brief summary follows.

Capper et al. [5], working with the effective theory in which Yang-Mills fields have only two physical components (hereafter referred to as LCM2), showed that with Mandelstam's prescription the self-energy has an infinite part that is manifestly renormalizable in the simple Yang-Mills theory and is ultraviolet-finite in the $N=4$ supersymmetric model. Leibbrandt and Matsuki [10] calculated the same quantities as Capper et al. but worked with versions of the theories in which all four components of the Yang-Mills field are retained (hereafter referred to as LCM4). They also used Leibbrandt's prescription instead of Mandelstam's. Their results are somewhat surprising: the self-energy has anomalous infinite parts that make the simple Yang-Mills theory appear to be unrenormalizable, and the $N=4$ model not ultraviolet-finite. Lee and Milgram [11] working with both the Mandelstam and the Leibbrandt prescriptions computed the Yang-Mills self-energies in both LCM2 and LCM4. They pointed out that the two prescriptions are equivalent and used the joint prescription to derive a representation for the complete class of two-point light-cone invariant integrals with which they resolved the apparent contradiction in the results of Capper et al. and Leibbrandt and Matsuki by showing that the self-energies of LCM4 and LCM2, although superficially different, are identical to $O(g^2)$ when the LCM4 result is translated into LCM2.

This paper is the sequel to ref. [11] written with two goals in mind. The first is to complete the one-loop study of the light-cone gauge for both LCM2 and LCM4.

Because of all the surprises and confusion the light-cone gauge has created in the past, and because some of the crucial properties of the Mandelstam-Leibbrandt prescription are not tested in the calculation of the one-loop self-energy, we feel it is important to remove all uncertainties concerning the gauge once and for all by computing the complete one-loop counter lagrangian. For LCM2 this is done by computing the one-loop corrections to the self-energy and the three- and four-vertices. The result is simple: the prescription works well, the counterterm associated with each Green function is proportional to the bare Green function and the complete counter lagrangian contains a single renormalization constant. The case for LCM4 is more complicated: the counterterms associated with the two and three-point Green functions are not proportional to the respective bare Green functions. However, when the two counterterms are summed, the anomalous terms cancel, so that the counter lagrangian is again characterized by one and the same renormalization constant as in LCM2. We have omitted computing the one-loop four-vertex in LCM4, partly because the calculation would be formidably lengthy, and partly because what we have calculated already determines uniquely what the counterterm associated with the four-vertex must be.

Our second goal is to demonstrate the utility of both light-cone invariance and the representation that was derived [11] for the class of two-point, light-cone invariant integrals. Our calculation is based on the method of analytic regularization described elsewhere [12]. We exploit the property of light-cone invariance in such a way that all the tensor integrals encountered in our calculation can be evaluated in terms of a single representation for a generalized Feynman integral. In pursuing this goal we demonstrate that LCM2, notwithstanding its unusual Feynman rules, is an exceedingly simple and easy gauge to work with, more so in our opinion than the simplest covariant gauges, at least at the one-loop level. In contrast, the bizarre renormalization property and the characteristically lengthy calculation dictates that LCM4 should be avoided.

For readers wishing to go directly to the results: see tables 4, 6 and 7 for the one-loop two, three and four-vertices in LCM2, respectively; for the same in LCM4 see tables 5, 8 and (5.14). For the representation of light-cone invariant integrals and two useful simplifying cases see (3.7) and (A.16) and (A.17). For the reduction of tensor integrals in LCM2 see table 2 and for the same in LCM4 see table 3.

In sect. 2 we present the notation and give the Feynman rules for LCM2 and LCM4. Apart from our own conventions, the content of this section is not new. In sect. 3 we discuss salient features of the Mandelstam and Leibbrandt prescriptions, give the representation for light-cone invariant two-point integrals (derived in appendix A) and show how the representation can also be used to evaluate light-cone-covariant and Lorentz-covariant tensor integrals. More details on this topic are given in appendix B. The calculation of the one-loop vertex functions is described and results are presented in sect. 4. Sect. 5 discusses the counterterms and renormalizability of LCM2 and LCM4. In sect. 6 we show that Slavnov-Taylor identities are satisfied in LCM4. These identities do not exist in LCM2 because gauge invariance is explicitly broken in the effective LCM2 lagrangian. Sect. 7 is the conclusion.

2. Notation and conventions

2.1. LIGHT-CONE COORDINATES

In Minkowski space with metric $(1, -1, -1, -1)$ the two light-cone null vectors have components

$$n_{\pm} \equiv \sqrt{\tfrac{1}{2}}\,[1,0,0,\pm 1], \tag{2.1}$$

satisfying

$$(n_+)^2 = (n_-)^2 = 0. \tag{2.2}$$

The normalizing factor is chosen such that

$$n_+ \cdot n_- = 1. \tag{2.3}$$

The light-cone components of any vector a are

$$a^{\pm} = a \cdot n_{\pm}. \tag{2.4}$$

A scalar product appears in light-cone coordinates as

$$a \cdot b = a_\mu b^\mu = a^+ b^- + a^- b^+ - \hat{a} \cdot \hat{b}, \tag{2.5}$$

where the caret denotes a two component subvector living on the $(1,2)$ plane

$$\hat{a} = [0, a_1, a_2, 0]. \tag{2.6}$$

By light-cone invariance we mean invariance under rotation confined to the $(1,2)$ plane. Thus the scalar product of caretted vectors

$$\hat{a} \cdot \hat{b} = a^i b^i, \tag{2.7}$$

where the summation of i over 1 and 2 is understood, is light-cone invariant. So are light-cone components such as $a^{\pm}$ and $b^{\pm}$. All Lorentz invariants are also light-cone invariants, whereas the inverse is not true.

When used as indices, the middle Latin letters $i, j, \ldots$ label the components of caretted vectors, the middle Greek letters $\mu, \nu \ldots$ label full Lorentz vectors and early latin letters $a, b, \ldots$ label vectors of the gauge group.

2.2. FEYNMAN RULES FOR LCM4

The gauge-fixing lagrangian corresponding to the constraint (1.1) for the light-cone gauge is

$$\mathcal{L}_{\mathrm{g.f.}} = -\lim_{\alpha\to 0}\frac{1}{2\alpha}(\underline{A}^{+})^{2}, \tag{2.8}$$

with the limit to be taken after the derivation of Feynman rules. The Faddeev-Popov ghosts being decoupled, the effective lagrangian is therefore

$$\mathcal{L}_{\mathrm{LCM4}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{g.f.}} = \tfrac{1}{2}\left[\underline{A}_{\mu}\circ(\partial^{2}\underline{A}^{\mu}) + (\partial_{\mu}\underline{A}^{\mu})^{2}\right]_{1}^{(2)} - \lim_{\alpha\to 0}\frac{1}{2\alpha}(\underline{A}^{+})^{2}$$
$$-g\left[(\partial_{\mu}\underline{A}_{\nu})\circ(\underline{A}^{\mu}\times\underline{A}^{\nu})\right]_{1}^{(3)} - \tfrac{1}{4}g^{2}\left[(\underline{A}_{\mu}\times\underline{A}_{\nu})^{2}\right]^{(4)}, \tag{2.9}$$

where the brackets are suffixed for later reference; $\underline{A}$ denotes a vector of the gauge group; $\circ$ and $\times$ refer to the inner and outer products $\underline{A}\circ\underline{B} = A^{a}B^{a}$ and $(\underline{A}\times\underline{B})^{a} = f^{abc}A^{b}B^{c}$, respectively. Feynman rules for LCM4 derived from (2.9) are given in table 1.

2.3. FEYNMAN RULES FOR LCM2

The implementation of the gauge constraint in LCM2 sets it apart from LCM4, and indeed from all other axial and covariant R_{ξ} gauges. Here the constraint (1.1) and the Euler-Lagrange equation (1.2) (derived from (2.9)) are used to eliminate the light-cone components $\underline{A}^{\pm}$ from $\mathcal{L}_{\mathrm{YM}}$ to give an effective lagrangian

$$\mathcal{L}_{\mathrm{LCM2}} = -\tfrac{1}{2}T^{(2)} + g\left(-T_{1}^{(3)} + T_{2}^{(3)}\right) - g^{2}\left(\tfrac{1}{4}T_{1}^{(4)} + \tfrac{1}{2}T_{2}^{(4)}\right), \tag{2.10}$$

where

$$T^{(2)} = \underline{A}^{i}\circ(\partial^{2}\underline{A}^{i}), \tag{2.11a}$$

$$T_{1}^{(3)} = (\partial^{j}\underline{A}^{i})\circ(\underline{A}^{j}\times\underline{A}^{i}), \tag{2.11b}$$

$$T_{2}^{(3)} = -(\partial^{i}\underline{A}^{i})\circ\left[(\partial^{+})^{-1}(\underline{A}^{j}\times\partial^{+}\underline{A}^{j})\right], \tag{2.11c}$$

$$T_{1}^{(4)} = (\underline{A}^{i}\times\underline{A}^{j})^{2}, \tag{2.11d}$$

$$T_{2}^{(4)} = \left[(\partial^{+})^{-1}(\underline{A}^{i}\times\partial^{+}\underline{A}^{i})\right]^{2}. \tag{2.11e}$$

For completeness we give the transformation from the terms in $\mathcal{L}_{\mathrm{LCM4}}$ to

TABLE 1
Feynman rules for light-cone gauge

(a) General ($\alpha, \beta, \ldots$ stand for $\mu, \nu, \ldots$ or $i, j, \ldots$)

Name	Diagram	Structure
propagator	——	$\Delta^{(0)ab}_{\alpha\beta}(p) = i\delta^{ab}\Delta^{(0)}_{\alpha\beta}(p)$
3-vertex		$\Gamma^{(0)abc}_{\alpha_1\alpha_2\alpha_3}(p_1, p_2, p_3) = gf^{abc}$ $[\tilde{\Gamma}^{(0)}_{\alpha_1\alpha_2\alpha_3}(p_1, p_2, p_3) + 2$ cyclic permutations$]$
4-vertex		$\Gamma^{(0)a_1a_2a_3a_4}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4) = ig^2\{[f^{a_1a_2b}f^{a_3a_4b}$ $\times \tilde{\Gamma}^{(0)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(p_1, p_2, p_3, p_4)] + [2 \leftrightarrow 3] + [2 \leftrightarrow 4]\}$

(b) LCM2

$$\Delta^{(0)}_{ij}(p) = -p^{-2}\delta_{ij}$$

$$\tilde{\Gamma}^{(0)}_{ijk}(p, q, r) = \delta_{ij}[(p-q)_k - r_k(p-q)^+/r^+], \qquad r \neq 0$$
$$= 2\delta_{ij}p_k, \qquad r = 0$$

$$\tilde{\Gamma}^{(0)}_{ijkl}(p, q, r, s) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{ij}\delta_{kl}\frac{(p-q)^+(r-s)^+}{[(p+q)^+(r+s)^+]}, \qquad r \neq -s$$
$$= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \qquad r = -s$$

(c) LCM4

$$\Delta^{(0)}_{\mu\nu}(p) = p^{-2}[g_{\mu\nu} - (p_\mu n_{+\nu} + p_\nu n_{+\mu})/p^+]$$
$$\tilde{\Gamma}^{(0)}_{\lambda\mu\nu}(p, q, r) = \delta_{\lambda\mu}(p-q)_\nu$$
$$\tilde{\Gamma}^{(0)}_{\lambda\mu\rho\sigma}(p, q, r, s) = \delta_{\lambda\rho}\delta_{\mu\sigma} - \delta_{\lambda\sigma}\delta_{\mu\rho}$$

those in $\mathcal{L}_{\mathrm{LCM2}}$

$$[\ ]^{(2)}_1 = -T^{(2)} + g^2T^{(4)}_2, \tag{2.12a}$$

$$[\ ]^{(3)}_1 = T^{(3)}_1 - T^{(3)}_2 + gT^{(4)}_2, \tag{2.12b}$$

$$[\ ]^{(4)} = T^{(4)}_1. \tag{2.12c}$$

The inverse transformation is not unique. It is significant that these transformations are not homogeneous in g.

Note that only $\hat{A}$ appears in (2.10), so LCM2 is an effective theory for a two-component field in the four-dimension Minkowski space. This fact has a very important bearing on the simplicity of computation in LCM2. Also, because both conditions (1.1) and (1.2) are realized in (2.10) so that no excess degrees of freedom associated with gauge invariance remain, the usual set of Slavnov-Taylor identities cannot be derived from LCM2.

Feynman rules derived from (2.10) are given in table 1. The ominous factor $1/p^+$ has shifted from the propagator in LCM4 to the three- and four-vertices in LCM2, distinguishing the latter from all other gauges. Note in particular the rules for $\Gamma^{(0)}_{ijk}(p,q,r)$ when $r=0$ and for $\Gamma^{(0)}_{ijkl}(p,q,r,s)$ when $r+s=0$, which at the one-loop level is tested only in calculations of the three- and four-vertices, but not in the self-energy [5,11] and the effective potential [13] calculations.

3. The generalized two-point integrals

3.1. THE MANDELSTAM-LEIBBRANDT PRESCRIPTION

In Minkowski space the nonpositivity of the propagator $1/q^2$ requires the prescription $\lim_{\eta\to 0^+} 1/(q^2+i\eta)$ to make the integrals well defined. The central problem in the light-cone gauge has been to find a compatible prescription for the factor $1/q^+$.

The Mandelstam [8] prescription is defined by

$$1/q^+ \to 1/[q^+]_{\mathrm{M}} \equiv \lim_{\eta\to 0^+} \left(q^+ + i\eta q^-\right)^{-1} \tag{3.1}$$

and the Leibbrandt [6] prescription is

$$1/q^+ \to 1/[q^+]_{\mathrm{L}} \equiv \lim_{\eta\to 0^+} q^-/(q^+q^- + i\eta)\,. \tag{3.2}$$

The reason that these prescriptions obey power counting, but the principal value prescription [14]

$$1/q^+ \to 1/[q^+]_{\mathrm{PV}} \equiv \lim_{\eta\to 0} q^+/\left[(q^+)^2+\eta^2\right] \tag{3.3}$$

does not, can be understood by comparing the poles of $[q^+]^{-1}_{\mathrm{M,L,PV}}$ and those of $(q^2+i\eta)^{-1}$ in the complex q_0 plane. The poles of $(q^2+i\eta)^{-1}$ and $[q^+]^{-1}_{\mathrm{M,L}}$ all have solutions characterized by

$$\operatorname{Im} q_0 \sim -\eta\, \operatorname{sign}[\operatorname{Re}(q_0)]\,, \tag{3.4}$$

whereas the poles of $[q^+]^{-1}_{\mathrm{PV}}$ occur at

$$\operatorname{Im} q_0 \sim \pm\eta\,, \tag{3.5}$$

regardless of the sign of $\operatorname{Re}(q_0)$. Now power counting is a property of integrals that can be defined in euclidean space, which for integrals originally defined in

Minkowski space is reached by continuing q_0 from the real axis to the imaginary axis. Since light-cone gauge integrals contain factors of $(q^+)^{-1}$ as well as $(q^2)^{-1}$, these factors must have the same analytic property in the q_0 plane insofar as continuation to euclidean space is concerned.

In appendix A it is shown that Leibbrandt's prescription is not distinct from Mandelstam's (at least for two-point integrals), and the two prescriptions will be referred to jointly as the Mandelstam-Leibbrandt prescription.

3.2. REPRESENTATION FOR A GENERALIZED TWO-POINT INTEGRAL

In appendix A it is shown that the class of generalized "two-point" integrals (so called because they involve one external momentum p)

$$M \equiv M(\omega, \kappa, \mu, \nu, \lambda; p) \equiv \int \mathrm{d}^{2\omega} q \left[(p-q)^2\right]^{\kappa} (q^2)^{\mu} (q^+)^{\nu} (q^-)^{\lambda}, \quad (3.6)$$

where ω, κ, μ, ν and $\lambda \geqslant 0$ are continuous real variables has the closed-form representation [11]

$$M = M_0 \frac{\Gamma(1+\lambda)}{\Gamma(-\nu)} \begin{cases} z^{-\nu} G^{2,3}_{3,3}\left(z \middle| {}^{1+\nu,1+\alpha_0,1+\alpha_1;}_{0,\beta_1;\beta_2}\right), & |z| \leqslant 1 \\ G^{3,2}_{3,3}\left(z^{-1} \middle| {}^{1+\nu,1-\beta_1+\nu;1-\beta_2+\nu}_{0,-\alpha_0+\nu,-\alpha_1+\nu;}\right), & |z| \geqslant 1, \end{cases} \quad (3.7)$$

where

$$\beta_1 \equiv \omega + \kappa + \nu, \qquad \alpha_1 \equiv \beta_1 + \mu, \qquad \alpha_0 \equiv -\omega - \mu - \lambda, \qquad \beta_2 = \nu - \lambda, \quad (3.8)$$

$$z \equiv 2p^+p^-/p^2, \quad (3.9)$$

$$M_0 = i(\pi \mathrm{e}^{-i\pi})^{\omega} (p^2)^{\alpha_1 - \nu} (p^+)^{\nu} (p^-)^{\lambda} / \Gamma(-\kappa)\Gamma(-\mu)\Gamma(\beta_1 - \alpha_0) \quad (3.10)$$

and the G's are Meijer G-functions [15], which are well-defined analytic functions easy to evaluate algebraically, either by hand or by machine.

The properties of the right-hand side of (3.7) have been described before; see appendix A. Suffice it to say here that it has poles and only poles in ω reflecting the ultraviolet and infrared singularities, that these poles are single and analytically separable [16] and that the integral obeys power counting. We emphasize that all tensor integrals needed to compute the complete one-loop counter lagrangian are reducible to M-integrals, (3.7).

3.3. TENSOR INTEGRALS

When computing the one-loop corrections to the two-, three- and four-point functions, it is necessary to evaluate two-point tensor integrals of the form

$$\underline{I}(p) = \int \mathrm{d}^4 q K(p,q) \underline{Q} \quad (3.11)$$

TABLE 2
Expansions for tensor integrals in LCM2

Tensor		Expansion
q_i	$\sim$	yp_i
q_iq_j	$\sim$	$(x-y^2)\hat{p}^2\delta_{ij}+(-x+2y^2)p_ip_j$
$q_iq_jq_k$	$\sim$	$y(x-y^2)\hat{p}^2(\delta_{ij}p_k+2$ symmetric terms) $+y(-3x+4y^2)p_ip_jp_k$
$q_iq_jq_kq_l$	$\sim$	$\frac{1}{3}(x-y^2)\hat{p}^4(\delta_{ij}\delta_{kl}+2$ terms) $-\frac{1}{3}(x-y^2)(x-4y^2)\hat{p}^2(p_ip_j\delta_{kl}+5$ terms) $+(x^2-8xy^2+8y^4)p_ip_jp_kp_l$

The sign " $\sim$ " means equivalent under integration; $x\equiv\hat{q}^2/\hat{p}^2$, $y\equiv\hat{p}\cdot\hat{q}/\hat{p}^2$.

where K is a light-cone invariant function and $\underline{Q}$ is a symmetric rank-n tensor having the form

$$\underline{Q}=q_{i_1}\cdots q_{i_n} \tag{3.12}$$

in LCM2 or the form

$$\underline{Q}=q_{\mu_1}\cdots q_{\mu_n} \tag{3.13}$$

in LCM4. It is shown in detail in appendix B that the tensor integral admits the substitution

$$\underline{Q}\sim\sum_l A_l\underline{S}^{(l)}, \tag{3.14}$$

$$A_l=\sum_{l'}\underline{Q}\cdot\underline{S}^{(l)}U_{l'l}, \tag{3.15}$$

where $\underline{S}^{(l)}$ is a set of symmetric tensors independent of q, and U is a symmetric matrix independent of the kernel K. The evaluation of any two-point tensor integral is thus reduced to that of a set of invariant integrals having the canonical form (3.6).

The expansions for tensors of up to rank-4 in LCM2 are exceedingly simple and are given in table 2. Corresponding expansions for LCM4, given in table 3 for tensors of rank 3 or less are considerably more lengthy; for each tensor the expansion is given in terms of the tensor set $\underline{S}^{(l)}$ and the matrix U.

The contrast between tables 2 and 3 is a clear indication of the difference in complexity that typifies computations in LCM2 and LCM4, respectively.

TABLE 3
Expansions for tensor integrals in LCM4

$\underline{Q} = q_\mu$

l	$\underline{S}^{(l)}$
1	p_μ
2	r_μ
3	s_μ

$$U = \frac{1}{\zeta p^2}\begin{bmatrix} -1 & 1 & 1 \\ & -1 & 1-2/z \\ & & -1 \end{bmatrix}$$

$\underline{Q} = q_\mu q_\nu$

l	$\underline{S}^{(l)}$	l	$\underline{S}^{(l)}$
1	$p^2 g_{\mu\nu}$	5	$p_\mu p_\nu$
2	$2(p_\mu r_\nu + p_\nu r_\mu)/z$	6	$2 r_\mu r_\nu / z$
3	$2(p_\mu s_\nu + p_\nu s_\mu)/z$	7	$2 s_\mu s_\nu / z$
4	$2(r_\mu s_\nu + r_\nu s_\mu)/z$		

$$U = \frac{1}{\zeta^2 p^4} \times \begin{bmatrix} \zeta^2 & -\alpha & -\alpha & \beta & \zeta & \alpha & \alpha \\ & \gamma & \delta & \varepsilon & -z & -\gamma & -\frac{1}{2}z \\ & & \gamma & \varepsilon & -z & -\frac{1}{2}z & -\gamma \\ & & & \phi & 1 & \psi & \psi \\ & & & & 2 & z & z \\ & & & & & \gamma & \chi \\ & & & & & & \gamma \end{bmatrix}$$

$\alpha = -z\frac{1}{2}\zeta$

$\beta = -\frac{1}{2}\zeta(z-2)$

$\gamma = \frac{1}{2}z^2, \qquad \delta = \frac{1}{4}z(z+1)$

$\varepsilon = \frac{1}{4}z(z-3), \qquad \phi = \frac{1}{2}(z^2 - 3z + 3)$

$\psi = \frac{1}{2} - z(z-2), \qquad \chi = \frac{1}{2}(z^2 - 2z + 2)$

$\underline{Q} = q_\lambda q_\mu q_\nu$

l	$\underline{S}^{(l)}$	l	$\underline{S}^{(l)}$
1	$p^2(g_{\lambda\mu} p_\nu$ + 2 symmetric terms)	7	$S^{(4)}(p \to s)$
2	$2p^2(g_{\lambda\mu} r_\nu$ + 2 terms)/z	8	$S^{(4)}(p \leftrightarrow r)$
3	$S^{(2)}(r \to s)$	9	$S^{(4)}(r \to p \to s)$
4	$2(p_\lambda p_\mu r_\nu$ + 2 terms)/z	10	$2(p_\lambda r_\mu s_\nu$ + 5 terms)/z
5	$S^{(4)}(r \to s)$	11	$p_\lambda p_\mu p_\nu$
6	$S^{(4)}(p \to r \to s)$	12	$2 r_\lambda r_\mu r_\nu / z$
		13	$2 s_\lambda s_\mu s_\nu / z$

$U = \frac{1}{3\zeta^3 p^6} \times$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$-\zeta^2$	α	α	β	β	A	A	$-\beta$	$-\beta$	B	-3ζ	β	β
2		γ	C	$-\delta$	ε	D	E	δ	ϕ	ϕ	β	$-\delta$	D
3			γ	ε	$-\delta$	E	D	ϕ	δ	ϕ	β	D	$-\delta$
4				$-\psi$	zQ	F	G	ψ	$-zG$	H	$6z$	$-\psi$	$-3z$
5					$-\psi$	G	F	$-zG$	ψ	H	$6z$	$-3z$	$-\psi$
6						I	J	K	L	M	6	N	P
7							I	L	K	M	6	P	N
8								$-\psi$	Q	R	$-6z$	ψ	3
9									$-\psi$	R	$-6z$	3	ψ
10										S	T	V	V
11											-12	$6z$	$6z$
12												$-\psi$	W
13													$-\psi$

$\alpha = \frac{1}{2} z\zeta^2$
$\beta = \frac{3}{2} z\zeta$
$\gamma = -\frac{1}{4} z^2\zeta^2$
$\delta = \frac{3}{4} z^2\zeta$
$\varepsilon = \frac{1}{4} z\zeta G$
$A = \frac{1}{2}(4 - z)\zeta$
$C = \frac{1}{4} z\zeta^2(z - 2)$
$E = -2(1 - z + \frac{3}{8}z^2)\zeta$
$G = -(z + 2)$
$I = -10 + 10z - 3z^2$
$K = z(5 - 2z)$
$N = 3z(z - 2)$

$R = \frac{1}{2} z(z - 7)$
$V = \frac{3}{2} z(3 - z)$
$\phi = -\frac{1}{4}\zeta F$
$\psi = 3z^2$
$B = \frac{1}{2}\zeta G$
$D = 3C/\zeta$
$F = z(z - 4)$
$H = \frac{1}{2} z(z + 5)$
$J = (z - 2)(5 - 5z + 3z^2)/z$
$L = 5 - 3z + z^2$
$P = -3(3 - 3z + z^2)$
$S = \frac{1}{2}(z^2 - 4z - 3)$
$W = 3(z - 2)(1 - z + z^2)/z$

$M = \frac{1}{2}(10 - 5z + z^2)$
$Q = -1 - 2z$
$T = -3(1 + z)$

$r \equiv p^- n_+$, $s \equiv p^+ n_-$, $z = 2p^+ p^-/p^2$, $\zeta \equiv z - 1$, U is symmetric; also see subsect. 3.3 and appendix B for notation.

4. The one-loop calculation

4.1. GENERAL REMARKS

We discuss the calculation of the one-loop, two-, three- and four-point vertices in LCM2 and the two- and three-point vertices in LCM4. In order to determine the counterterms needed to carry out the renormalization program, it is sufficient to examine vertices with one nonvanishing external momentum.

The general algorithm used in our calculation follows. For an n-point vertex the calculation involves tensor integrals of rank n or less. These will be evaluated using the method described in subsect. 3.3. The vertex, with one external momentum p, has the expansion

$$\underline{\Gamma}(p) = \sum_l F_l(z, \varepsilon) \underline{Q}^{(l)}, \tag{4.1}$$

where $\underline{\Gamma}$ is an n-point vertex which is itself a rank-n tensor, $\underline{Q}^{(l)}$ is the set of L_n rank-n tensors carrying the symmetry of $\underline{\Gamma}$ and F_l are light-cone invariant functions expressible as linear sums of M-integrals or, as explicit functions of z ($\equiv 2p^+p^-/p^2$) and $\varepsilon = \omega - 2$ after the integrals are evaluated using (3.7). When $n > 2$, the tensor set

$\underline{O}^{(l)}$ is larger than the symmetric set $\underline{S}^{(l)}$ of subsect. 3.3, but is constructed from the same basis tensor set (see appendix B) that spans $\underline{S}^{(l)}$. For LCM2, $L_2 = 2$, $L_3 = 3$ and $L_4 = 10$; for LCM4, $L_2 = 7$ and $L_3 = 24$.

The only important ε-dependent terms in F_l are of $O(1/\varepsilon)$, which reflect either the ultraviolet or the infrared divergence in the vertex function. These divergences are easily separated analytically with (3.7). In our calculation, the only infrared divergence was found in the four-point vertex. Therefore, unless otherwise specified, poles of $O(1/\varepsilon)$ in our result are of ultraviolet origin. Most of the poles originate from M-integrals with $\nu \geqslant 0$, in which case the representation reduces to a terminating ${}_3F_2$ hypergeometric function (see (A.17)).

Normally it is considerably more difficult to evaluate the finite (i.e., $O(\varepsilon^0)$) part of a Green function than the infinite part. Again because of (3.7), that task is straightforward in our computation. When

$$\kappa + \mu + \nu \leqslant -3, \qquad \nu < 0, \qquad \lambda \geqslant 0, \tag{4.2}$$

the integral becomes an infinite series which, for $|z| \leqslant 1$, we write as

$$\begin{aligned} M_\infty(\kappa, \mu, \nu, \lambda; z) &\equiv M(\kappa, \mu, \nu, \lambda; p)/[M_0 \Gamma(1+\lambda)/\Gamma(-\nu)] \\ &= (-)^{1+\beta_1} z^{-\nu} \sum_{l=0} z^l \left\{ \prod_i [\Gamma(l-\alpha_i)/\Gamma(1+l-\beta_i)] \right\} \\ &\quad \times \left\{ \ln z + \sum_i [\psi(l-\alpha_i) - \psi(1+l-\beta_i)] \right\}, \end{aligned} \tag{4.3}$$

where i runs from 0 to 2, $\alpha_2 \equiv \nu$, $\beta_0 \equiv 0$, and the other parameters are defined in (3.8) and (3.9). In the calculation we have encountered repeatedly three distinct series, associated with the M_∞ integrals thus

$$M_\infty(-1,-1,-1,0;z) \equiv -S_1(z), \tag{4.4}$$

$$\begin{aligned} M_\infty(-1,-1,-1,1;z) &= M_\infty(-1,-2,-1,1;z) \\ &= -M_\infty(-2,-1,-1,1;z) \equiv -S_2(z), \end{aligned} \tag{4.5}$$

$$\begin{aligned} M_\infty(-1,-3,-1,2;z) &= M_\infty(-3,-1,-1,2;z) \\ &= -M_\infty(-2,-2,-1,2;z) \equiv -S_3(z), \end{aligned} \tag{4.6}$$

with

$$S_n(z) = \sum_{l=0} \frac{z^{l+1}}{l+n} \left(\ln z - \frac{1}{l+n} \right). \tag{4.7}$$

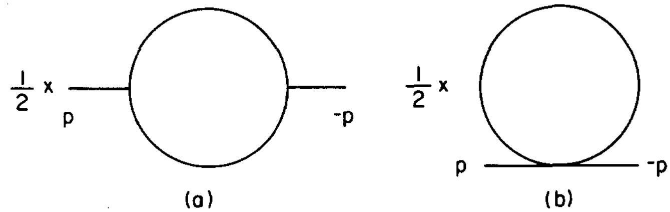

Fig. 1. Two diagrams for the self-energy. The tadpole, (b), is zero-valued in LCM4, but not in LCM2.

The relation (4.4) has previously been obtained [5,11]. From (4.7), one finds the reductions

$$S_2 = 1 - \ln z + S_1/z, \tag{4.8a}$$

$$S_3 = \tfrac{1}{4} - \tfrac{1}{2}\ln z + (1 - \ln z)/z + S_1/z^2 \tag{4.8b}$$

so only S_1 need appear in the result. The computation was done with the algebraic computer code SCHOONSCHIP [17].

4.2. THE SELF-ENERGY

The self-energies (fig. 1) for both LCM2 [5,11] and LCM4 [10,11] have been calculated previously, but for completeness we repeat them here. The infinite parts are

$$(\Pi_{\mathrm{LCM2}})_{ij}^{ab}|_{\mathrm{infinite}} = i\delta^{ab}g^2 Z_\varepsilon \tfrac{11}{3} p^2 \delta_{ij}, \tag{4.9}$$

$$(\Pi_{\mathrm{LCM4}})_{\mu\nu}^{ab}|_{\mathrm{infinite}} = i\delta^{ab}g^2 Z_\varepsilon \Big[-\tfrac{11}{3}\big(p^2 g_{\mu\nu} - p_\mu p_\nu\big) + 2p^+\big(p_\mu n_{-\nu} + p_\nu n_{-\mu}\big) + \text{terms dep. on } n_+ \Big], \tag{4.10}$$

with

$$Z_\varepsilon \equiv C_2/16\pi^2\varepsilon, \qquad \varepsilon \equiv \omega - 2. \tag{4.11}$$

The n_+ dependent terms in Π_{LCM4} are unimportant because they cannot contribute to the counterterm – see (1.1). The full expressions for the self-energies are given in tables 4 and 5, respectively.

We make the following remarks.

(i) All the infinite parts in (4.9,10) are associated with ultraviolet (UV) divergences.

(ii) In LCM4 the tadpole diagram, fig. 1b, is exactly zero.

(iii) In LCM2 the tadpole diagram is nonzero and contributes to both the infinite and finite parts.

TABLE 4
One-loop self-energy in LCM2

$$\Pi_{ij}^{ab}(p) = i\delta^{ab}\frac{g^2 C_2}{16\pi^2}\sum_{l=1} F_l O^{(l)}$$

$$O^{(1)} = p^2\delta_{ij}, \qquad O^{(2)} = p_i p_j$$

$$F_1 = \frac{11}{3e_1} - \tfrac{70}{9} - 4S_1, \qquad F_2 = \frac{2}{3(z-1)}$$

Notation: $e_1^{-1} = \varepsilon^{-1} + \ln(p^2\pi) + \gamma - i\pi$; $z = 2p^+p^-/p^2$; $S_1 = \sum_{l=1} \frac{z^l}{l}\left(\ln z - \frac{1}{l}\right)$.

TABLE 5
One-loop self-energy in LCM4

$$\Pi_{\mu\nu}^{ab}(p) = i\delta^{ab}\frac{g^2 C_2}{16\pi^2}\sum_{l=1}^{7} F_l O^{(l)}$$

l	$O^{(l)}$	l	$O^{(l)}$
1	$p^2 g_{\mu\nu}$	5	$r_\mu r_\nu$
2	$p_\mu p_\nu$	6	$s_\mu s_\nu$
3	$p_\mu r_\nu + p_\nu r_\mu$	7	$r_\mu s_\nu + r_\nu s_\mu$
4	$p_\mu s_\nu + p_\nu s_\mu$		

$$F_1 = -\frac{11}{3e_1} + \tfrac{70}{9} + 4S_1$$

$$F_2 = \frac{11}{3e_1} - \tfrac{64}{9} + \frac{1}{z-1}\left(\frac{2}{3} + 4z\ln z\right)$$

$$F_3 = \frac{2}{e_1} - 4 - \frac{1}{z-1}\left[\frac{2}{3} + (8-2z)\ln z\right] - \tfrac{8}{3}S_1$$

$$F_4 = -\frac{2}{e_1} + 4 - \frac{1}{z-1}\left(\frac{2}{3} + 2z\ln z\right)$$

$$F_5 = -\frac{8}{ze_1} + \frac{1}{z(z-1)}\left[\frac{50}{3}z - 16 + (16-8z)\ln z\right] + \frac{1}{z^2}S_1$$

$$F_6 = \frac{2}{3(z-1)}$$

$$F_7 = \frac{4}{ze_1} + \frac{1}{z(z-1)}\left(\tfrac{28}{3} - \tfrac{26}{3}z + 4z\ln z\right)$$

Notation: $r \equiv p^- n_+$; $s \equiv p^+ n_-$; see also table 4.

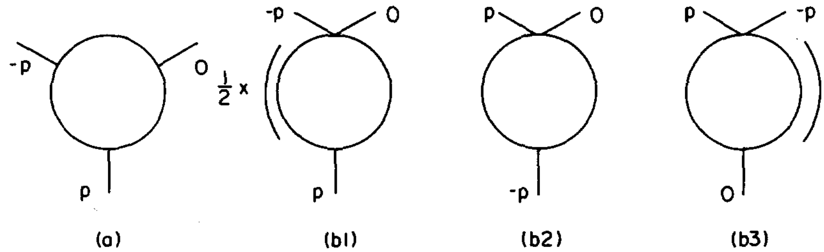

Fig. 2. Four diagrams for the three-vertex $\Gamma(p, -p, \mathrm{o})$. The "tadpole", (b3), is algebraically zero.

(iv) The second term in the square bracket in (4.10) is anomalous, in the sense that the counterterm, $(\partial_\mu \underline{A}^\mu)\cdot(\partial^+ \underline{A}^-)$, needed to cancel it is not contained in the unrenormalized $\mathcal{L}_{\mathrm{LCM4}}$, suggesting that in LCM4 the two-point Green function by itself may not be renormalizable. However, we will see in sect. 5 that $\mathcal{L}_{\mathrm{LCM4}}$ is renormalizable. Note that the LCM2 self-energy does not contain anomalous infinite parts.

4.3. THE THREE-POINT VERTEX IN LCM2

We compute only the special case $\Gamma^{abc}_{ijk}(p, -p, \mathrm{o})$ (see fig. 2), since its infinite part is sufficient to determine the relevant counterterm. In this calculation the Feynman rules given in table 1 for the three- and four-vertices involving the expression $(p^+)^{-1}$ in the limit $p \to \mathrm{o}$ are tested for the first time. The Mandelstam-Leibbrandt prescription for this limit is

$$\lim_{p\to \mathrm{o}} (1/p^+) \to \lim_{p\to \mathrm{o}} (1/[p^+]_{\mathrm{ML}}) = 0, \tag{4.12}$$

giving the results in table 1b.

The calculated infinite part of the one-loop three-vertex is

$$\Gamma^{abc}_{ijk}(p, -p, \mathrm{o})|_{\mathrm{infinite}} = g^3 f^{abc} Z_\varepsilon \left(-\tfrac{22}{3}\right)\delta_{ij} p_k, \tag{4.13}$$

exactly as required for renormalizability. The complete result is given in table 6. Note that the invariant function F_2 vanishes; in fact the contributions to F_2 from fig. 2a, b vanish separately. Furthermore, in contrast to the tadpole fig. 1b in the case of the self-energy, the tadpole fig. 2(b3) is zero.

4.4. THE FOUR-POINT VERTEX IN LCM2

Again, to determine the counterterm it is sufficient to calculate the special case $\Gamma^{abcd}_{ijkl}(p, -p, \mathrm{o}, \mathrm{o})$. Even for this simplification the vertex is considerably more complicated than the three-vertex. The twelve diagrams to be calculated are shown

TABLE 6
One-loop three-vertex in LCM2

$$\Gamma_{ijk}^{abc}(p,-p,0)=gf^{abc}\frac{g^2C_2}{16\pi^2}\sum_{l=1}^{3}F_lO^{(l)}$$

$O^{(1)}=\delta_{ij}p_k$, $O^{(2)}=\delta_{jk}p_i+\delta_{ki}p_j$, $O^{(3)}=p_ip_jp_k/\hat{p}^2$

$F_1=\frac{-22}{3e_1}+\frac{74}{9}+\frac{8}{z-1}z\ln z+8S_1$

$F_2=0$, $F_3=-\frac{4}{3}$.

Notation: see table 4.

in fig. 3. There are ten rank-4 tensors (see table 7) and three independent gauge-group couplings:

$$d_1\equiv C_2^{-1}\mathrm{Tr}(t^at^bt^ct^d), \tag{4.14a}$$

$$d_2\equiv C_2^{-1}\mathrm{Tr}(t^at^ct^bt^d), \tag{4.14b}$$

$$d_3\equiv C_2^{-1}\mathrm{Tr}(t^at^bt^dt^c), \tag{4.14c}$$

where the matrix t^a has elements

$$(t^a)_{bc}=f^{abc}. \tag{4.15}$$

The couplings that appear in the bare vertex (see table 1)

$$f^{abe}f^{cde}\equiv f_1=2(d_3-d_1), \tag{4.16a}$$

$$f^{ace}f^{bde}\equiv f_2=2(d_3-d_2), \tag{4.16b}$$

$$f^{ade}f^{cbe}\equiv f_3=2(d_2-d_1), \tag{4.16c}$$

can be expressed in terms of the d_n's, but the reverse is not true. Taking into account all the tensors and group couplings, the four-vertex altogether has 30 invariant functions,

$$\Gamma_{ijkl}^{abcd}(p,-p,0,0)=\sum_{n,m}F_{n,m}d_nO_{ijkl}^{(m)}. \tag{4.17}$$

Because the vertex is symmetric under exchange of any two external lines, all the invariant functions are not independent, but satisfy the relations

$$F_{1,m}=F_{3,m'},\qquad F_{2,m}=F_{2,m'} \tag{4.18}$$

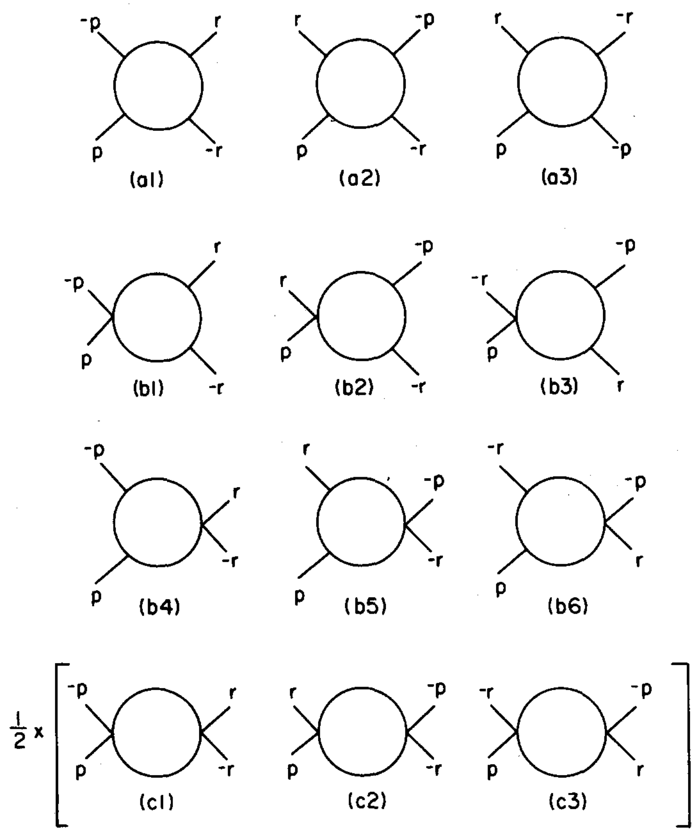

Fig. 3. Twelve diagrams for the four-vertex. In the calculation $r=0$, and the diagrams were not separately evaluated.

for the pairs

$$(m, m') = (1,1), (2,3), (4,4), (5,6), (7,8), (9,9), (10,10), \qquad (4.19)$$

resulting in only 17 independent invariant functions.

The complete result is given in table 7. The infinite parts are

$$\Gamma^{abcd}_{ijkl}(p, -p, 0, 0)|_{\text{infinite}} = -ig^2\{4g^2 Z'_\varepsilon f_1(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \tfrac{11}{3}g^2 Z_\varepsilon[2f_1(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (f_2 - f_3)\delta_{ij}\delta_{kl}]\}, \qquad (4.20)$$

where Z'_ε is the same as Z_ε except that it originates from IR divergence. Thus only those terms in (4.20) proportional to Z_ε need be cancelled by counterterms.

TABLE 7
One-loop four-vertex in LCM2

$$\Gamma^{abcd}_{ijkl}(p,-p,0,0) = ig^2 \frac{g^2 C_2}{16\pi^2}(\Gamma_{\rm inf} + \Gamma_{\rm reg}); \qquad \Gamma_{\rm reg} = \sum_{n=1}^{3} \sum_{m=1}^{10} d_n F'_{n,m} O^{(m)}$$

m	$O^{(m)}$	m	$O^{(m)}$	m	$O^{(m)}$
1	$\delta_{ij}\delta_{kl}$	5	$p_i p_k \delta_{jl}/\hat{p}^2$	8	$p_l p_j \delta_{ki}/\hat{p}^2$
2	$\delta_{ik}\delta_{jl}$	6	$p_i p_l \delta_{kj}/\hat{p}^2$	9	$p_k p_l \delta_{ij}/\hat{p}^2$
3	$\delta_{il}\delta_{kj}$	7	$p_k p_j \delta_{il}/\hat{p}^2$	10	$p_i p_j p_k p_l/\hat{p}^4$
4	$p_i p_j \delta_{kl}/\hat{p}^2$				

$$\Gamma_{\rm inf} = -\frac{11}{3e_1}[2f_1(O_2 - O_3) + (f_2 - f_3)O_1] - \frac{4}{e_{IR}} f_1(O_2 - O_3); \qquad O_m \equiv O^{(m)}$$

$$F'_{1,1} = F'_{3,1} = -8z + \frac{46}{3} + \frac{8}{z-1} z \ln z + 8S_1$$

$$\left.\begin{matrix} F'_{1,2} = F'_{3,3} \\ F'_{1,3} = F'_{3,2} \end{matrix}\right\} = \mp\left(\tfrac{166}{9} + 8\ln z + 16 S_1\right) + \tfrac{16}{9}$$

$$F'_{1,4} = F'_{3,4} = -\tfrac{28}{9}, \qquad F'_{1,6} = F'_{1,7} = F'_{3,5} = F'_{3,8} = -\tfrac{40}{9}$$

$$F'_{1,5} = F'_{1,8} = F'_{3,6} = F'_{3,7} = \tfrac{8}{9}, \qquad F'_{1,10} = F'_{2,10} = F'_{3,10} = \tfrac{16}{3}$$

$$F'_{1,9} = F'_{3,9} = \frac{116}{3} z - \frac{196}{9} - \frac{16}{z-1} z^2 \ln z$$

$$F'_{2,1} = -\frac{52}{3} - \frac{16}{z-1} z \ln z - 16 S_1$$

$$F'_{2,2} = F'_{2,3} = 2F'_{2,4} = -F'_{2,5} = -F'_{2,6} = -F'_{2,7} = -F'_{2,8} = \tfrac{16}{9}$$

$$F'_{2,9} = -\frac{184}{3} z + \frac{272}{9} + \frac{32}{z-1} z^2 \ln z$$

Notation: see subsect. 4.4 and table 4.

4.5. THE THREE-POINT VERTEX IN LCM4

The computation of the vertex $\Gamma^{abc}_{\lambda\mu\nu}(p,-p,0)$ which involves rank-3 tensor integrals requires the use of the 13×13 U-matrix, table 3.

Because the tadpole fig. 2(b3) is zero algebraically and

$$\Gamma^{abc}_{\lambda\mu\nu}(p,-p,0)|_{(b1)} = \Gamma^{abc}_{\lambda\mu\nu}(p,-p,0)|_{(b2)}, \tag{4.21}$$

only the diagram fig. 3a and fig. 3(b1) need be evaluated. The relevant infinite parts are

$$\Gamma^{abc}_{\lambda\mu\nu}(p,-p,0) = g^3 f^{abc} Z_\varepsilon\Big[-\tfrac{11}{3}(2g_{\lambda\mu}p_\nu - g_{\mu\nu}p_\lambda - g_{\nu\lambda}p_\mu) + 2p^+(2g_{\lambda\mu}n_{-\nu} - g_{\mu\nu}n_{-\lambda} - g_{\nu\lambda}n_{-\mu}) + \text{terms dep. on } n_+\Big], \tag{4.22}$$

with the full expression given in table 8. The n_- dependent infinite part requires an

TABLE 8
One-loop three-vertex in LCM4

$$\Gamma^{abc}_{\lambda\mu\nu}(p,-p,0)=gf^{abc}\frac{g^2C_2}{16\pi^2}(\Gamma_{\rm int}+\Gamma_{\rm reg});\qquad \Gamma_{\rm reg}=\sum_{l=1}^{24}F_l'O^{(l)}$$

l	$O^{(l)}$	l	$O^{(l)}$	l	$O^{(l)}$
1	$g_{\lambda\mu}p_\nu$	9,10	$O^{(7,8)}(r\to s)^\dagger$	20	$O^{(19)}(r\to s\to p\to r)$
2	$g_{\mu\nu}p_\lambda+g_{\nu\lambda}p_\mu$	11,12	$O^{(7,8)}(p\leftrightarrow r)$	21	$O^{(19)}(r\leftarrow s\leftarrow p\leftarrow r)$
3,4	$O^{(1,2)}(p\to r)$	13,14	$O^{(7,8)}(r\to p\to s)$	22	$p_\lambda p_\mu p_\nu/p^2$
5,6	$O^{(1,2)}(p\to s)$	15,16	$O^{(7,8)}(p\to s)$	23	$r_\lambda r_\mu r_\nu/p^2$
7	$r_\lambda r_\mu p_\nu/p^2$	17,18	$O^{(7,8)}(p\to r\to s)$	24	$s_\lambda s_\mu s_\nu/p^2$
8	$(r_\lambda p_\mu+r_\mu p_\lambda)r_\nu/p^2$	19	$(r_\lambda s_\mu+r_\mu s_\lambda)p_\nu/p^2$		

$$\Gamma_{\rm inf}=\frac{1}{e_1}\Big[-\tfrac{11}{3}(2O_1-O_2)+2O_4+2(2O_5-O_6)$$
$$+\frac{1}{z}(-16O_7+8O_{19}-4O_{20}-4O_{21})+\frac{16}{z^2}O_{23}\Big];\qquad O_l\equiv O^{(l)}$$
$$F_1'=\tfrac{74}{9}+\frac{8}{z-1}z\ln z+8S_1$$
$$F_2'=-\tfrac{70}{9}+\frac{4}{z-1}z\ln z$$
$$F_5'=-\frac{4}{z-1}(z+1)\ln z$$
$$F_6'=4-\frac{2}{z-1}z\ln z$$

Notation: See tables 4 and 5. Finite parts of other invariant functions are not given here.
† In all cases the normalization factor p^2 is not changed in permutations.

anomalous counterterm for its cancellation. However, as will be shown shortly, all anomalous counterterms cancel in the total counter lagrangian for LCM4.

The finite part for the three-vertex is too lengthy to be quoted in full. Instead, in table 8 we give only the finite parts of those operators appearing in (4.22).

5. Counterterms and renormalizability

5.1. COUNTERTERMS IN LCM2

The counterterms that will cancel the infinite parts in the self-energy (4.9), the three-point vertex (4.13) and the four-point vertex (4.20) respectively are

$$\delta\mathcal{L}^{(2)}_{\rm LCM2}=-\tfrac{1}{2}(Z_3-1)T^{(2)},\tag{5.1}$$

$$\delta\mathcal{L}^{(3)}_{\rm LCM2}=(Z_1-1)g\big(-T_1^{(3)}+T_2^{(3)}\big),\tag{5.2}$$

$$\delta\mathcal{L}^{(4)}_{\rm LCM2}=-(Z_4-1)g^2\big(\tfrac{1}{4}T_1^{(4)}+\tfrac{1}{2}T_2^{(4)}\big),\tag{5.3}$$

with

$$Z_3 = Z_1 = Z_4 \equiv Z = 1 + \tfrac{11}{3} g^2 Z_\varepsilon , \tag{5.4}$$

where $Z_\varepsilon \equiv C_2/16\pi^2\varepsilon$ as before. This shows that LCM2 is multiplicatively renormalizable order-by-order in powers of g. The identity of all three renormalizations also ensures that the "Ward identity"

$$Z_1^2 = Z_3 Z_4 \tag{5.5}$$

is satisfied.

From (5.4), the β-function [18] of the renormalization group is

$$\beta(g) = 2g\left(Z_1^{-1} Z_3^{3/2}\right)\left[\partial\left(Z_1 Z_3^{-3/2}\right)/\partial(1/\varepsilon)\right]$$
$$= -\tfrac{11}{3} Z_\varepsilon g^3 + \mathrm{O}(g^5) , \tag{5.6}$$

which, as expected, is identical to that calculated in covariant gauges [9].

Summing (5.1)–(5.3) and comparing with (2.10), we find

$$\delta\mathcal{L}_{\mathrm{LCM2}} = (Z-1)\mathcal{L}_{\mathrm{LCM2}} , \tag{5.7}$$

which has the characteristic simplicity, typical of ghost-free gauges, of having a single renormalization constant. In such gauges the complete (Yang-Mills) counter lagrangian and the β-function can be determined directly by the self-energy. In contrast, because of the presence of ghost fields, in covariant gauges the complete counter lagrangian contains six renormalization constants [19] (related by one Ward identity). Another gauge known to have the property (5.7) is the axial gauge ($n^2 \neq 0$) [3], which however is typified by computations [16] much more complex and lengthy than those in LCM2.

Of course it has long been suspected that the light-cone gauge would have only one renormalization constant. As far as we know, until now the technique for evaluating light-cone gauge integrals has not been sufficiently mastered for this belief to be verified. We now show that the belief is true in an unexpectedly subtle way in the case of LCM4.

5.2. COUNTERTERMS IN LCM4

The counterterms required to cancel the (UV) infinite parts in the self-energy (4.10) and the three-point vertex (4.22) are respectively

$$\delta\mathcal{L}^{(2)}_{\mathrm{LCM4}} = \tfrac{1}{2}(Z-1)[\;]^{(2)}_1 + Y\left[\left(\partial_\mu \underline{A}^\mu\right)\circ\left(\partial^+ \underline{A}^-\right)\right]^{(2)}_2 , \tag{5.8}$$

$$\delta\mathcal{L}^{(3)}_{\mathrm{LCM4}} = -(Z-1)g[\;]^{(3)}_1 - Yg\left[\left(\partial^+ \underline{A}_\mu\right)\circ\left(\underline{A}^- \times \underline{A}^\mu\right)\right]^{(3)}_2 , \tag{5.9}$$

where Z is defined in (5.4) and

$$Y = -2Z_\varepsilon g^2. \tag{5.10}$$

The anomalous counterterm $[\ ]_2^{(2)}$ is needed to cancel the term $p^+(p_\mu n_{-\nu} + p_\nu n_{-\mu})$ in (4.10) and $[\ ]_2^{(3)}$ is needed to cancel the term $p^+(2g_{\lambda\mu}n_{-\nu} - g_{\mu\nu}n_{-\lambda} - g_{\nu\lambda}n_{-\mu})$ in (4.22). The nonvanishing of the anomalous renormalization constant Y in (5.8) and (5.9) means that LCM4 is not renormalizable order-by-order in powers of g.

At first glance this appears to counter our proof in the last section that the light-cone gauge is renormalizable. The contradiction is resolved when one notes that the two terms $[\ ]_2^{(2)}$ and $[\ ]_2^{(3)}$ are actually identical in LCM2, although they possess different powers in $\underline{A}$ and g in LCM4. Thus, from (1.1) and (1.2)

$$[\ ]_2^{(2)} = g[\ ]_2^{(3)} = -gT_2^{(3)} + g^2 T_2^{(4)} \tag{5.11}$$

and the two anomalous terms cancel in the sum

$$\delta\mathcal{L}_{\text{LCM4}}^{(2+3)} = (Z-1)\left(\tfrac{1}{2}[\ ]_1^{(2)} - g[\ ]_1^{(3)}\right), \tag{5.12}$$

which, upon comparison with (2.9), has the desired form to $O(g^3)$.

Since LCM2 and LCM4 are different versions of the same theory, the difference in (5.7) and (5.12) uniquely determines the infinite part of the four-vertex in LCM4. For from (5.7), (5.12), (2.10) and (2.12)

$$\delta\mathcal{L}_{\text{LCM4}}^{(4)} = \delta\mathcal{L}_{\text{LCM2}} - \delta\mathcal{L}_{\text{LCM4}}^{(2+3)} = -\tfrac{1}{2}(Z-1)g^2[\ ]^{(4)} \tag{5.13}$$

and it follows that

$$\Gamma_{\kappa\lambda\mu\nu}^{abcd}(p,q,r,s)|_{\text{infinite}} = -\tfrac{11}{3}g^2 Z_\varepsilon \Gamma_{\kappa\lambda\mu\nu}^{(0)abcd}(p,q,r,s). \tag{5.14}$$

6. Slavnov-Taylor identities

In LCM4, because the gauge-fixing constraint (1.1) is satisfied by adding a gauge-fixing term to the effective lagrangian, the Ward-Takahashi-Slavnov-Taylor (Ward for short) identities [20] can be derived as usual, and the two that can be verified by quantities we have calculated are

$$p_\mu \Pi_{\mu\nu}(p) = 0, \tag{6.1}$$

$$p_\lambda \Gamma_{\lambda\mu\nu}^{abc}(p,-p,0) = igf^{abc}\Pi_{\mu\nu}(p). \tag{6.2}$$

From the results given in tables 5 and 8, it is seen that (6.1) is satisfied, as is the infinite part of (6.2). Furthermore, each of the identities give rise to a set of relations among invariant functions. With the short-hand notation $F_{n+m+\dots} = F_n + F_m + \dots$, from (6.1) the relations are

$$(2F_{1+2} + zF_{3+4})_\Pi = (2F_3 + zF_{5+7})_\Pi = (2F_4 + zF_{6+7})_\Pi = 0 \tag{6.3}$$

and from (6.2) they are

$$-(2F_1)_\Pi = (2F_2 + zF_{4+6})_\Gamma, \tag{6.4a}$$

$$-(2F_2)_\Pi = (2F_{1+2} + zF_{12+14+22})_\Gamma, \tag{6.4b}$$

$$-(2F_3)_\Pi = (2F_{3+11} + zF_{8+20})_\Gamma = (2F_{4+12} + zF_{7+19})_\Gamma, \tag{6.4c}$$

$$-(2F_4)_\Pi = (2F_{5+13} + zF_{10+21})_\Gamma = (2F_{6+14} + zF_{9+19})_\Gamma, \tag{6.4d}$$

$$-(2F_5)_\Pi = (2F_8 + zF_{16+23})_\Gamma, \tag{6.4e}$$

$$-(2F_6)_\Pi = (2F_{10} + zF_{18+24})_\Gamma, \tag{6.4f}$$

$$-(2F_7)_\Pi = (zF_{15+18} + 2F_{21})_\Gamma = (zF_{16+17} + 2F_{20})_\Gamma. \tag{6.4g}$$

We have verified that the *finite* parts as well as the infinite parts of all these relations are satisfied. Observe that knowledge of the infinite parts of the self-energy, the renormalizability conditions

$$\tfrac{1}{2}(F_1)_{\Gamma,\mathrm{inf}} = -(F_2)_{\Gamma,\mathrm{inf}} = (F_1)_{\Pi,\mathrm{inf}}, \tag{6.5a}$$

$$\tfrac{1}{2}(F_5)_{\Gamma,\mathrm{inf}} = -(F_6)_{\Gamma,\mathrm{inf}} = -(F_4)_{\Pi,\mathrm{inf}}, \tag{6.5b}$$

$$(F_{9,10,13,14,22,24})_{\Gamma,\mathrm{inf}} = 0 \tag{6.5c}$$

and the Ward identities (6.4a)–(6.4g) combined *does not* determine the infinite parts of the three-vertex.

In LCM2, the gauge-fixing constraint is not satisfied via the usual method of lagrangian multiplier. Rather the effective lagrangian is obtained by substituting the solutions of the constraints (1.1) and (1.2) into the Yang-Mills lagrangian, thus explicitly removing all redundant degrees of freedom associated with gauge invariance. Consequently Ward identities are lost to LCM2. This can also be understood by noting that n_+ dependent terms are absent in LCM2 but are crucial to the Ward identities in LCM4, even though such terms are irrelevant in the construction of counterterms.

7. Conclusion

We have thoroughly studied the two light-cone gauge theories LCM2 and LCM4 at the one-loop level. We have computed all the one-loop vertex functions of both theories except the four-point function in LCM4. We find that in LCM2, the infinite parts of all the one-loop vertex functions are proportional to the respective bare vertex functions, rendering LCM2 renormalizable order-by-order in the coupling constant g. Furthermore the counter lagrangian contains only one renormalization constant $Z = 1 + \frac{11}{3}g^2C_2/16\pi^2\varepsilon$.

The situation for LCM4 is more complicated. Both the one-loop self-energy and the three-vertex have anomalous infinite parts, requiring anomalous counterterms for their cancellation. However, although these counterterms are of $O(A^2)$ and $O(gA^3)$ respectively, they become identical and cancel each other exactly when transformed into LCM2 operators. The result is that the renormalizability of LCM4 does not become obvious until both the one-loop self-energy and the three-vertex are calculated, at which stage the combined counterterm is proportional to the bare lagrangian to $O(g^3)$, characterized again by the same single renormalization constant Z of LCM2. From this it is deduced that the renormalization constant associated with the four-vertex must also be Z – there is no freedom left for the appearance of any new anomalous counterterm.

Thus, as far as the renormalization property is concerned, the one-loop structure of LCM2 is completely determined by the infinite part of the one-loop self-energy. We emphasize that although it is commonly believed that this is generally true for all ghost-free gauges (it is true for the axial gauge), we have shown that the rule is applicable to LCM4, which is ghost-free and renormalizable, only with the provision that anomalous counterterms be ignored.

In this work we have demonstrated the usefulness of the analytic representation (3.7) for the class of generalized light-cone invariant two-point integrals; armed with that single representation the entire calculation was reduced to one involving only linear algebra. We have presented the finite as well as the infinite parts for most of the vertex functions calculated – the full result for the three-vertex in LCM4 is too long to be given here – so that they may be verified by anyone who wishes to do so by another method.

The method of analytic regularization preserves gauge invariance; in particular we have verified that the two- and three-point Slavnov-Taylor identities in LCM4 are satisfied. Because of the way gauge-fixing is implemented in LCM2, vertex functions in this theory are not known to satisfy any identities. The analytic method also separates IR from UV divergences. Among the vertex functions calculated, only the four-vertex in LCM2 has an IR infinite part.

This study has been carried out in greater detail than would be normally deemed necessary in order to demonstrate convincingly that the light-cone gauge, which has previously caused so much confusion and misunderstanding, is now well understood and, technically, completely under control. Because of its outstanding simplicity compared with any other gauge, we believe LCM2 is fully justified to be the gauge of choice. In comparison, LCM4 is a far less attractive formalism. Its similarity with LCM2 ends with the absence of ghosts. Calculations in LCM4 are considerably lengthier than corresponding calculations in LCM2. Equally undesirable is the fact that radiative corrections to vertex functions of LCM4 have anomalous infinite parts that are apt to cause confusion. We do not recommend the use of this version of the light-cone gauge.

We thank George Leibbrandt for useful communications.

Appendix A

We prove the equivalence of Mandelstam's and Leibbrandt's prescriptions and derive (3.7). The integration is carried out in Minkowski space; all variables are real and continuous.

First consider the integral

$$I_{\mathrm{M}}(a,b,c;\nu,\lambda) \equiv \lim_{\eta\to 0^+} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\,(x+i\eta y)^{\nu}(y+i\eta)^{\lambda} \times \exp[2i(ax+by+cxy)]\,, \qquad c,\lambda \geqslant 0\,. \tag{A.1}$$

The connection of this integral to Mandelstam's prescription is recognized when one reads $(x+i\eta y)^{\nu}$ as $(q^+ + i\eta q^-)^{\nu}$. Use Euler's formula

$$z^{\nu} = \frac{1}{\Gamma(-\nu)} \int_0^{\infty} \mathrm{d}t\, t^{-\nu-1} \mathrm{e}^{-zt}, \qquad \mathrm{Re}(z) > 0 \tag{A.2}$$

and write

$$\begin{aligned} x + i\eta y &= i(-ix+\eta y)\,, \qquad y \geqslant 0 \\ &= -i(ix-\eta y)\,, \qquad y < 0\,, \end{aligned} \tag{A.3}$$

so that

$$I_{\mathrm{M}} = \frac{i^{\lambda+\nu}}{\Gamma(-\nu)\Gamma(-\lambda)} \lim_{\eta\to 0^+} \int_0^{\infty} \mathrm{d}t\, t^{-\nu-1} \int_0^{\infty} \mathrm{d}s\, s^{-\lambda-1} \mathrm{e}^{-\eta s} \times \int_0^{\infty} \mathrm{e}^{-\eta y t}\mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}x\, \{\exp[i(2b+s)y + i(2a+cy+t)x] + (-)^{\nu}\exp[-i(2b+s)y + i(2a-cy-t)x]\}\,. \tag{A.4}$$

The integrations over x, y and s are trivial, yielding

$$I_M = \frac{\pi}{c\Gamma(-\nu)}\left(-\frac{a}{c}\right)^{\lambda}(2ia)^{-\nu}e^{-2iab/c}\int_0^1 dt\, t^{-\nu-1}(1-t)^{\lambda}e^{2iabt/c}. \quad (A.5)$$

The analog to I_M for Leibbrandt's prescription is

$$I_L \equiv \lim_{\eta\to 0^+}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\,(xy+i\eta)^{\nu}(y+i\eta)^{\lambda-\nu}\exp[2i(ax+by+cxy)]\,. \quad (A.6)$$

Because the imaginary part in the factor $(xy+i\eta)$ no longer depends on y, it is not necessary to split the y-integration into two parts as in (A.3). After using (A.2) and integrating over x, y and one of the Euler parameters one obtains

$$I_L = \frac{i^{\lambda}\pi}{c\Gamma(-\nu)}(2c)^{-\nu}\left(\frac{ia}{c}\right)^{\lambda-\nu}\int_0^{\infty} dt\, t^{-\nu-1}(1+t)^{-\lambda+\nu-1}e^{-2iab/[c(1+t)]}, \quad (A.7)$$

which readily transforms to the form of (A.5) when the integration variable t is changed to

$$\tau = t/(1+t)\,.$$

This shows that the Mandelstam and Leibbrandt prescriptions are identical. The subscripts on I will henceforth be dropped.

Now consider the M-integral

$$M(\omega,\kappa,\mu,\nu,\lambda) \equiv \int d^{2\omega}q\left[(p-q)^2\right]^{\kappa}(q^2)^{\mu}(q^+)^{\nu}(q^-)^{\lambda}, \qquad \lambda \geqslant 0\,. \quad (A.8)$$

Use Mandelstam's (or Leibbrandt's) prescription $q^+ \to q^+ + i\eta q^-$ for the factor q^+; use the usual η-prescription $z \to z + i\eta$ for the other three factors $(p-q)^2$, q^2 and q^-; write

$$\int d^{2\omega}q = \int d^{2(\omega-1)}\hat{q}\int_{-\infty}^{\infty} dq^+ \int_{-\infty}^{\infty} dq^-\,;$$

use

$$p\cdot q = p^+q^- + p^-q^+ - \hat{p}\cdot\hat{q}$$

and apply (A.2) to obtain

$$M = \lim_{\eta\to 0^+}\frac{i^{\kappa+\mu}}{\Gamma(-\kappa)\Gamma(-\mu)}\int_0^{\infty} dr \int_0^{\infty} ds\, r^{-\kappa-1}s^{-\mu-1}e^{-\eta(r+s)}e^{ip^2 r}$$

$$\times \int d^{2(\omega-1)}\hat{q}\, e^{i[-\hat{q}^2(r+s)+2r\hat{p}\cdot\hat{q}]}I(-p^- r, -p^+ r, r+s; \nu, \lambda)\,. \quad (A.9)$$

The method of dimensional regularization [21] gives

$$\int d^{2(\omega-1)}\hat{q}\, e^{i[\cdots]} = [-i\pi/(r+s)]^{\omega-1} e^{i\hat{p}^2 r^2/(r+s)}. \tag{A.10}$$

Use (A.5) for the I-integral, transform variables

$$r = \rho(1-\tau), \qquad s = \rho\tau \tag{A.11}$$

and integrate over ρ to obtain

$$\begin{aligned} M = {} & i(\pi e^{-i\pi})^{\omega}(p^2)^{\alpha_1-\nu}(p^+)^{\nu}(p^-)^{\lambda} z^{-\nu}\Gamma(-\alpha_1)[\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)]^{-1} \\ & \times\left\{\int_0^1 d\tau\, \tau^{\beta_1-1}(1-\tau)^{-1-\alpha_0}\int_0^1 dt\, t^{-\nu-1}(1-t)^{\lambda}\right. \\ & \left.\times[1+zt(1-\tau)/\tau]^{\alpha_1}\right\}, \end{aligned} \tag{A.12}$$

where $\alpha_0 = -\omega-\mu-\lambda$, $\alpha_1 = \omega+\kappa+\mu+\nu$, $\beta_1 = \omega+\kappa+\nu$, $z = 2p^+p^-/p^2$. The double integral has a known G-function representation [12] for $|z| \leqslant 1$,

$$\left\{\int_0^1 d\tau \int_0^1 dt \cdots\right\} = \frac{\Gamma(1+\lambda)}{\Gamma(-\alpha_1)\Gamma(\beta_1-\alpha_0)} G^{2,3}_{3,3}\left(z \,\middle|\, \begin{matrix} 1+\alpha_0, & 1+\alpha_1, & 1+\alpha_2; \\ 0, & \beta_1; & \beta_2 \end{matrix}\right), \tag{A.13}$$

where the parameters $\alpha_2 = \nu$ and $\beta_2 = \nu-\lambda$ are introduced to expose the full symmetry of the G-function. (A.13), when substituted into (A.12), gives the first representation of (3.7) in the text. The second representation, for $|z| \geqslant 1$, results from a known analytic continuation of the G-function [15].

The representation derived here is very similar to but distinct from the G-function representation derived in ref. [12] for axial gauge integrals, and can be evaluated using the same method described in detail there. A particularly useful expansion for the G-function in (A.13) for $|z| < 1$ is

$$\begin{aligned} G^{2,3}_{3,3}(z|\ldots) = {} & \frac{\Gamma(\beta_1)\Gamma(-\alpha_0)\Gamma(-\alpha_1)\Gamma(-\alpha_2)}{\Gamma(1-\beta_2)}\, {}_3F_2\left(\begin{matrix} -\alpha_0, -\alpha_1, -\alpha_2 \\ 1-\beta_1, 1-\beta_2 \end{matrix} \,\middle|\, z\right) \\ & + z^{\beta_1}\frac{\Gamma(-\beta_1)\Gamma(\beta_1-\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}{\Gamma(1-\alpha_0)} \\ & \times {}_3F_2\left(\begin{matrix} \beta_1-\alpha_0, \beta_1-\alpha_1, \beta_1-\alpha_2 \\ 1+\beta_1, 1-\alpha_0 \end{matrix} \,\middle|\, z\right), \end{aligned} \tag{A.14}$$

where ${}_3F_2$ is a hypergeometric function. A similar expansion [15] exists for the $G^{3,2}_{3,3}(z^{-1}|\cdots)$ in (3.7) for $|z| \geqslant 1$.

When (3.7) is used as a representation of Feynman integrals, the exponents κ, μ, ν and λ have integer values. The limiting process that preserves gauge invariance and facilitates the analytic separation of ultraviolet (UV) and infrared (IR) divergences in the Feynman integral is [16]

$$\kappa - \sigma, \mu - \sigma, \lambda, \nu = \text{integers}, \qquad \sigma \to 0, \qquad \omega = 2 + \varepsilon. \tag{A.15}$$

Gauge invariance is preserved only when $\sigma = 0$, in which case all divergences are of $O(1/\varepsilon)$. However, because the ultraviolet and infrared poles have different σ-dependence and therefore approach $1/\varepsilon$ differently as $\sigma \to 0$, the parameter can be used to tag and separate the two types of poles.

Four simplifying special cases of the representation are particularly useful.

(i) $\kappa \to$ integer $\geqslant 0$. M is either identically zero or is a sum of UV and IR poles with residues equal in magnitude but opposite in sign.

(ii) $\mu \to$ integer $\geqslant 0$.

$$M = L_0 \Gamma(\alpha_1 - \alpha_0)\Gamma(\alpha_1 - \alpha_2)\Gamma(-\alpha_1)\Gamma(1+\lambda)/[\Gamma(1+\alpha_1-\beta_2)\Gamma(-\nu)]$$

$$\times z^{\alpha_1 - \alpha_2} {}_3F_2\left(\begin{matrix} -\mu, -\alpha_1, -\alpha_1 + \beta_2 \\ 1 - \alpha_1 + \alpha_0, 1 - \alpha_1 + \alpha_2 \end{matrix} \middle| z^{-1} \right), \tag{A.16}$$

$$L_0 \equiv i(\pi e^{-i\pi})^{\omega} (p^2)^{\alpha_1 - \nu} (p^+)^{\nu} (p^-)^{\lambda} / [\Gamma(-\kappa)\Gamma(\beta_1 - \alpha_0)] .$$

All the parameters of the ${}_3F_2$ except μ depend on ω, therefore according to (A.15) the right-hand side of (A.16) is a terminating polynomial in powers of z^{-1} of at most $(\mu + 1)$ terms.

(iii) $\nu =$ integer $\geqslant 0$.

$$M = L_0 \Gamma(-\alpha_0 + \alpha_2)\Gamma(-\alpha_1 + \alpha_2)\Gamma(\beta_1 - \alpha_2)/\Gamma(-\mu)$$

$$\times {}_3F_2\left(\begin{matrix} -\nu, -\lambda, \beta_1 - \alpha_2 \\ 1 + \alpha_0 - \alpha_2, 1 + \alpha_1 - \alpha_2 \end{matrix} \middle| z^{-1} \right), \tag{A.17}$$

which is a polynomial in powers of z^{-1} of order ν or λ (recall λ is always $\geqslant 0$), whichever is less.

(iv) $\mu \to$ integer $\geqslant 0$ and $\nu =$ integer $\geqslant 0$. M is the same as in case (i). Cases (i) and (iv) combined form the generalized class of "tadpole" integrals, which are zero-valued when UV and IR divergences are not distinguished, as in dimensional regularization.

Note added in proof

We emphasize that the representation (3.7) is only valid for $\lambda \geqslant 0$. The apparent discrepancy between (3.6) ($\lambda \geqslant 0$) and (A.2) ($\nu < 0$) indicates that another prescription for q^- along the lines of (3.1) or (3.2) would be necessary for a rigorous extension to the case $\lambda < 0$. This is unnecessary for this work, and the quoted results are valid. We thank Q. Ho-Kim for pointing this out to us.

Appendix B

We describe the expansion of tensor integrals having the general form

$$\underline{I}(p) \equiv \int \mathrm{d}^{2\omega} q K(p, q) \underline{Q}, \tag{B.1}$$

where K is a function of light-cone invariant quantities such as $p \cdot q$, $\hat{p} \cdot \hat{q}$, q^+ and q^-, and $\underline{Q}$ is a symmetric tensor composed of factors of q_α's; α runs from 1 to 2 in LCM2 and from 0 to 3 in LCM4. For example, the rank-2 $\underline{Q}$ in LCM2 is $q_i q_j$ and the rank-3 $\underline{Q}$ in LCM4 is $q_\lambda q_\mu q_\nu$.

Since $\underline{I}$ carries the same indices as $\underline{Q}$, it is clear that in LCM2 $\underline{I}$ must transform as a symmetric tensor constructed from the vector p_i and the tensor δ_{ij}. We shall call (p_i, δ_{ij}) the basis tensor set for LCM2. Similarly, the basis tensor set for LCM4 is $(p_\mu, r_\mu, s_\mu, g_{\mu\nu})$, where for simplicity we define the two vectors

$$r \equiv p^- n_+ , \qquad s \equiv p^+ n_- . \tag{B.2}$$

The following discussion applies equally to LCM2 and LCM4. Suppose $\underline{Q}$ is rank-n. Then $\underline{I}$ admits the expansion

$$\underline{I} = A_l \underline{S}^{(l)}, \tag{B.3}$$

where summation over l, an index labelling all the independent rank-n symmetric tensors $\underline{S}^{(l)}$ that can be constructed from the basis tensor set, is understood. Define the inverse matrix U^{-1} with elements

$$(U^{-1})_{ll'} \equiv \underline{S}^{(l)} \cdot \underline{S}^{(l')} \tag{B.4}$$

and the set of light-cone invariant quantities

$$a_l \equiv \underline{Q} \cdot \underline{S}^{(l)}. \tag{B.5}$$

Then

$$A_l = \int \mathrm{d}^{2\omega} q K a_{l'} U_{l'l}, \tag{B.6}$$

$$\underline{I} = \underline{S}^{(l)} \int \mathrm{d}^{2\omega} q K a_{l'} U_{l'l} \tag{B.7}$$

since $\underline{S}^{(l)}$ does not depend on q. Thus the evaluation of the tensor integral $\underline{I}$ is reduced to the evaluation of a set of light-cone invariant integrals, which can be further reduced to M-integrals and evaluated using the representation (3.7).

In (B.7), the quantities a_l and $U_{ll'}$ are independent of the kernel K, therefore, by comparison with (B.1), one may write

$$\underline{Q} \sim a_{l'} U_{l'l} \underline{S}^{(l)}, \tag{B.8}$$

which is an equivalence true under the integration $\int d^{2\omega} q$ for any light-cone invariant kernel.

The complete expansion (B.8) for tensors up to rank 4 in LCM2 are given in table 2. Because of the larger basis tensor set, corresponding expansions in LCM4 are considerably lengthier. For tensors of rank-3 or less the expansions are given in table 3 in terms of U and $\underline{S}^{(l)}$.

The method described here is readily generalized to deal with tensor integrals in covariant gauges, where the basis tensor set is $(p_\mu, g_{\mu\nu})$, and axial gauges, where the basis tensor set is $(p_\mu, n_\mu, g_{\mu\nu})$. A table of the most commonly used integrals in the light-cone gauge is available [22].

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