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Flat connections on quantum bundles and fractional statistics in geometric quantization

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It is shown that in geometric quantization formulation, fractional statistics in a quantized system of N indistinguishable particles in two spatial dimensions arise from the nontrivial cohomology of the flat connection on the quantum line bundles as well as from the nontrivial homology of the configuration space. The propagator of a nonrelativistic interacting system with fractional statistics is derived.

I. INTRODUCTION

The possible existence of fractional statistics in $(2+1)$ dimensions has been recognized and discussed for many years.¹ It has been suggested that fractional statistics may be relevant to important physical phenomena such as the fractional quantum Hall effect² and high T_c superconductivity.³ Since the unusual statistical property in two dimensions is closely related to the nontrivial topology, namely, the global property, of their configuration space of a system of indistinguishable particles,^{1,4} and since the geometric quantization formulation^{5,6} is essentially a globalization of canonical quantization, we think it is natural and important to discuss fractional statistics in the context of this formulation. This paper is devoted to this subject.

In the framework that permits the existence of Fermi-Dirac statistics, the configuration space of N indistinguishable particles in d spatial dimensions is⁴

$$M^{dN} \equiv (R^{dN} - \mathcal{D})/S_N, \quad (1)$$

where $R^{dN} = (R^d)^{\otimes N}$ is the dN -dimensional Euclidean space, $\mathcal{D} = \{(q_1 \dots q_N) \mid q_i = q_j, i \neq j = 1, \dots, N\}$ is the set of diagonal points, and S_N is the permutation group. The fundamental group $\pi_1(M^{dN})$ of M^{dN} depends decisively on d . For $d \geq 3$ it is S_N and for $d=2$ it is Artin's braid group B_N (M^{dN} is disconnected when $d=1$). As is well-known, the difference between B_N and S_N is responsible for the radically dissimilar statistical properties of systems in $d \geq 3$ and in $d=2$ spaces. We find that in the geometric quantization formulation the curial factor giving rise to quantum statistics is the flat part of the connection on the quantum line bundle. When the Hilbert space is chosen to be composed of covariantly constant sections along the vertical polarization, that is, when the Schrödinger position representation is selected, a nontrivial choice of the cohomology class of the flat part of the connection combines with the homology of configuration space in $d=2$ dimensions to give unusual statistics.

In Sec. II the method of Refs. 5 and 6 is used to give a review of the classical geometric description of an N particle system and to quantize it. In Sec. III the relation between statistics and the aforementioned connection on the complex line bundle is examined. In Sec. IV the

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statistical phase factor is expressed in terms of a nontrivial solution to the equation for the flat connection, and the Blattner–Konstant–Sternberg (BKS) kernel is used to derive a path-integral expression for the propagator of a nonrelativistic system. Section V contains our conclusion.

II. GEOMETRIC QUANTIZATION OF A SYSTEM OF N INDISTINGUISHABLE PARTICLES

Following the method of Refs. 5 and 6, we first construct a symplectic geometry for a system of N indistinguishable particles, whose configuration space M^{dN} is given by Eq. (1) and whose phase space $\Gamma = T^*M^{dN}$, is the cotangent bundle over M^{dN} . The symplectic manifold (Γ, ω) is the manifold Γ equipped with a nondegenerate closed two-form ω (called the symplectic form) on Γ . That is

$$d\omega = 0 \quad (2)$$

and nondegenerate

$$\omega(X, \bullet) = 0 \Rightarrow X = 0, \quad (3)$$

where X is a tangent vector field on Γ .

Associated with each smooth function f , which may be a classical observable on Γ , we define a tangent vector field X_f called the Hamiltonian vector field by

$$X_f \lrcorner \omega = -df, \quad (4)$$

where $\lrcorner$ denotes the inner product. Define the Poisson bracket of two functions f and g as

$$\{f, g\} \equiv -\omega(X_f, X_g) = -X_f g = X_g f. \quad (5)$$

The closure of ω ensures the Jacobi identity on the Poisson bracket. Let γ be a map of the interval $(0, 1)$ to M^{dN} : $(0, 1) \rightarrow M^{dN}$. If f is the Hamiltonian function, then the equation for the integral curve $\gamma(t)$ of X_f is

$$\frac{d}{dt} \gamma(t) = X_f \gamma(t) \quad (6)$$

is the canonical Hamiltonian equation of motion.

Because the symplectic form ω is closed, it can be expressed locally as an exact form

$$\omega = d\theta, \quad (7)$$

where the canonical one-form symplectic potential θ is defined only up to a closed one-form. Also ω and θ may be expressed locally in terms of the canonical local coordinates (q_i^a, p_{ia}) , $i = 1, \dots, N$, $a = 1, \dots, d$

$$\begin{aligned} \omega &= \sum_{i,a} dp_{ia} \wedge dq_i^a, \\ \theta &= \sum_{i,a} p_{ia} dq_i^a. \end{aligned} \quad (8)$$

Being dependent only on ω , classical mechanics is invariant under general canonical transformations which always preserves ω .

The first step in geometric quantization is to associate a linear operator $\mathcal{P}_f$ with each function f on Γ such that $\mathcal{P}_1$ is the identity operator and all $\mathcal{P}_f$ satisfy the commutation relation

$$[\mathcal{P}_f, \mathcal{P}_g] = i\hbar \mathcal{P}_{\{f, g\}}. \quad (9)$$

This may be achieved by introducing a complex line bundle L , called the prequantization line bundle, over Γ . Denote by L^x the bundle obtained from L by removing its zero section; this is the $U(1)$ principle bundle over Γ associated with L . On L^x define a connection one-form α and a curvature two-form Ω related by

$$\Omega = d\alpha. \quad (10)$$

Denote by η_c the fundamental vector field on L^x and by $\eta_c(z)$ the tangent vector of the curve $t \mapsto e^{2\pi i t} z$, where z is the coordinate on a fiber in L^x . Then α satisfies

$$\eta_c \lrcorner \alpha = c, \quad (11)$$

where $c \in \mathbb{C}$ is a complex number. From Eqs. (10) and (11)

$$\eta_c \lrcorner \Omega = 0, \quad \mathcal{L}_{\eta_c} d\alpha = 0, \quad (12)$$

where $\mathcal{L}_{\eta_c}$ is the Lie derivative along η_c . Hence, Ω is the pull back of a closed two-form on Γ . Thus the general form of α is

$$\alpha = \delta + \beta, \quad (13)$$

where β is a pull back of a one-form on Γ and, according to Eq. (11), δ admits a local expression

$$\delta = dz/2\pi iz. \quad (14)$$

The operators $\mathcal{P}_f$ act on the space of sections of the bundle L as follows. The one-parameter group of canonical transformations ϕ_f^t generated by f has a unique lift to a one-parameter group of connection preserving transformations of the bundle, which defines the action of ϕ_f^t on the sections. The operator $\mathcal{P}_f$ is defined by

$$\mathcal{P}_f \lambda = i\hbar \frac{d}{dt} (\phi_f^t \lambda) \big|_{t=0} = (-i\hbar \nabla_{X_f} + f) \lambda, \quad (15)$$

$$\nabla_{X_f} = X_f + 2\pi i X_f \lrcorner \beta,$$

where λ is a section of L and ∇_{X_f} is the covariant derivative along the direction X_f . Condition (9) requires

$$\Omega = -\hbar^{-1} \omega, \quad (16)$$

which, from Eqs. (7) and (10), implies that in general

$$\beta = -\hbar^{-1} \theta - \alpha_0, \quad (17)$$

where α_0 is an arbitrary closed one-form,

$$d\alpha_0 = 0. \quad (18)$$

Since α_0 does not contribute to Ω , we call α_0 the flat part of the connection one-form.

The prequantization space, or the space of sections of L , forms the full representation space of quantum physics. However, this space is obviously too big to be the correct physical Hilbert

space, since locally a section admits functions of both q_i^a and p_{ia} , which would lead to violation of the uncertainty principle. In order to prevent this from happening it is necessary to reduce by "polarization" the prequantization space to a suitable subspace.

A polarization $\mathcal{F}$ of a symplectic manifold (Γ, ω) is an involutive complex distribution on Γ satisfying

$$\dim_{\mathbb{C}} \mathcal{F} = \frac{1}{2} \dim \Gamma, \quad \omega|_{\mathcal{F} \times \mathcal{F}} = 0. \quad (19)$$

Given a polarization $\mathcal{F}$, the space of sections of L can be restricted to a subspace of sections that are covariantly constant along $\mathcal{F}$, the complex conjugate distribution of $\mathcal{F}$. The conjugate $\bar{\psi}$ of such a section ψ must be covariantly constant along $\mathcal{F}$. In general, the Hermitian product $\psi\bar{\psi}'$ is covariantly constant along $D = \mathcal{F} \cap \bar{\mathcal{F}}$ and its integral over Γ diverges unless the leaves of D are compact. This may suggest that it should be natural to integrate $\psi\bar{\psi}'$ over Γ/D , except that there is not natural measure on Γ/D . One way to circumvent this difficulty is to use half forms to construct the correct density to be integrated over Γ/D . A completely self-contained description of half forms is lengthy; interested readers are referred to Refs. 5 and 6. Here, we only give a very simple presentation.

Let $\mathcal{F}$ be a polarization on Γ which forms an n -dimensional vector space, $\{X_a\}$ an arbitrary basis of $\mathcal{F}$ whose linear transformations form the group $GL(n; \mathbb{C})$, and G a matrix representation of an element in $GL(n; \mathbb{C})$. Denote by $\delta_r(\mathcal{F})$ the set of all functions v on $\mathcal{F}$ with the property

$$v\{(XG)_a\} = (\det G)^r v\{X_a\}. \quad (20)$$

The elements in $\delta_r(\mathcal{F})$ may be thought of as the r th power of the volume element on $\mathcal{F}$. Every $\delta_r(\mathcal{F})$ is a one-dimensional complex vector space and the vector spaces of $\delta_r(\mathcal{F})$ satisfy

$$\begin{aligned} \delta_0(\mathcal{F}) &= \mathcal{C}, \quad \delta_1(\mathcal{F}) = \wedge^{\otimes n} \mathcal{F}^*, \\ \delta_r(\mathcal{F}) &= (\delta_{-r}(\mathcal{F}))^* = \delta_{-r}(\mathcal{F}^*), \quad \delta_r(\bar{\mathcal{F}}) = \overline{\delta_r(\mathcal{F})}, \\ \delta_r(\mathcal{F}) \otimes \delta_s(\mathcal{F}) &= \delta_{r+s}(\mathcal{F}), \end{aligned} \quad (21)$$

where $\mathcal{C}$ denotes constant functions, $*$ denotes dual, and overline denotes complex conjugation. It can be shown that the bundle over Γ whose sections give the correct quantum description of physical state is $L \otimes \delta_{-1/2}(\bar{\mathcal{F}})$. Then the polarized wave functions are locally of the form

$$\psi = s\nu, \quad (22)$$

where s is a (polarized) section in L satisfying

$$\nabla_X s = 0, \quad \forall X \in \{\bar{\mathcal{F}}\}, \quad (23)$$

and ν is a section of $\delta_{-1/2}(\bar{\mathcal{F}})$ satisfying

$$\nabla_X \nu = 0, \quad \forall X \in \{\bar{\mathcal{F}}\}. \quad (24)$$

The sections of $\delta_{-1/2}(\bar{\mathcal{F}})$ are called half forms. The inner product is defined as

$$\langle \psi | \psi \rangle = \int_{\Gamma/D} s_m \bar{s}_m |\bar{\nu}_m|^2 |\epsilon_{\omega'}|^{1/2} |\epsilon_{\omega}|, \quad (25)$$

where the subscript m denotes a point in Γ/D ,

$$\epsilon_\omega = (-1)^{n(n-1)/2} \frac{1}{n!} \omega^n \quad (26)$$

is the Liouville form on Γ , and $\epsilon_{\omega'}$ is the Liouville form on $(\mathcal{F}_m + \bar{\mathcal{F}}_m)/D_m$.

The above description is applicable to any polarization. In this paper we focus on the Schrödinger's position picture, i.e., we take the vertical polarization in which $\mathcal{F}$ is spanned by the linear frame fields $\{\partial/\partial p_i^a; i=1, \dots, N; a=1, \dots, d\}$. Then the polarized wave functions have the form

$$\psi = \psi(q_i^a) \hat{s} v, \quad (27)$$

where $\hat{s}$ is the unit section. Since $\Gamma/D = M^{dN}$, the inner product can be reduced to

$$\int \psi(q) \bar{\psi}'(q) |dq_1^1 \wedge \dots \wedge dq_N^d|. \quad (28)$$

The reduction of the full prequantization representation space to the Hilbert space $\mathcal{H}$ of the sections of $L \otimes \delta_{-1/2}(\bar{\mathcal{F}})$, which is covariantly constant along $\bar{\mathcal{F}}$, requires that the definition (15) of operators be modified. Consider a function f on Γ and its Hamiltonian vector field X_f . Denote by ϕ_f^t the one-parameter group of canonical transformations generated by X_f with parameter t ; ϕ_f^t is a map on the prequantization representation space. For each $t \in \mathbb{R}$, the image of the polarization $\mathcal{F}$ of Γ under the derived mapping $\mathcal{T} \phi_f^t$ is a polarization $\mathcal{F}_t \equiv \mathcal{T} \phi_f^t(\mathcal{F})$ of the symplectic (Γ, ω) . For each $\psi \in \mathcal{H}$, one can define $\phi_f^t \psi$. This gives a map $\phi_f^t: \mathcal{H} \rightarrow \mathcal{H}_t$ whose target space $\mathcal{H}_t$ consists of covariantly constant sections of $L \otimes \delta_{-1/2}(\bar{\mathcal{F}})$ along $\bar{\mathcal{F}}_t$. If f is a polarization-preserving function, i.e., if $\mathcal{F}_t = \mathcal{F}$, then a quantum operator $\mathcal{O}_f$ on $\mathcal{H}$ can be defined as in Eq. (15)

$$\mathcal{O}_f(\psi) = i\hbar \frac{d}{dt} (\phi_f^t \psi) |_{t=0}, \quad \psi \in \mathcal{H}. \quad (29)$$

On the other hand, if f is not a polarization-preserving function, i.e., if $\mathcal{F}_t \neq \mathcal{F}$, then (see Sec. 5 in Ref. 6) there exists a Blattner-Konstant-Sternberg (BKS) kernel $\mathcal{K}_t: \mathcal{H}_t \times \mathcal{H} \rightarrow \mathbb{C}$ and a linear map $\mathcal{U}_t: \mathcal{H}_t \rightarrow \mathcal{H}$ such that

$$\mathcal{K}_t(\psi, \psi_t) = \langle \psi | \mathcal{U}_t \psi_t \rangle. \quad (30)$$

For small $t \in (0, \epsilon)$ define $\phi_t: \mathcal{H} \rightarrow \mathcal{H}_t$ by

$$\phi_t = \mathcal{U}_t \circ \phi_f^t. \quad (31)$$

Then the quantum operator is given by

$$\mathcal{O}_f = i\hbar \frac{d}{dt} \phi_t |_{t=0}. \quad (32)$$

III. QUANTUM CONNECTION AND STATISTICS

In the last section it was shown that in a given classical system the symplectic two-form ω is fixed, but in the quantized system the quantum connection one-form α is determined only up to a flat connection one-form α_0 . Different choices of α_0 correspond to the same classical limit but may give distinct quantum systems. In this section we will show that, for a system of N indistinguishable particles. A choice of cohomology class to which α_0 belongs to is precisely the choice of the statistics of the quantum system. Often statistics is thought of as just the exchange

symmetry of the wave functions of indistinguishable particles. However, as is well-known,³ the issue is not just exchange symmetry when M^{dN} is not simply connected. To see this, let $\gamma(t)$ be a curve in M^{dN} as before. Since M^{dN} is base manifold of the cotangent bundle Γ it is a submanifold of the base manifold of the quantization bundle $\hat{L}$. After having been fixed to the vertical polarization the Hilbert space $\mathcal{H}$ may be considered as a bundle over M^{dN} . Then, because the representation space is restricted to the Schrödinger's position representation,

$$\omega(X_i X_j) = 0, \quad \forall X_i X_j \in V(M^{dN}). \quad (33)$$

Therefore, from Eq. (16), the curvature Ω vanishes, and the connection one-form α restricted to M^{dN} is flat, or $d\alpha = 0$ on M^{dN} . This means that α on M^{dN} may be classified into equivalence classes according to the first de Rham cohomology $H^1(M^{dN})$ of M^{dN} .

The parallel transport of the wave function ψ along the curve $\gamma(t)$ on M^{dN} is given by the solution to the equation for the horizontal lift of $\gamma(t)$,

$$\nabla_{X_{\gamma(t)}} \psi(\gamma(t)) = 0, \quad (34)$$

where $X_{\gamma(t)}$ is the tangent vector field of $\gamma(t)$. The equation has the formal solution

$$\psi(\gamma(t)) = \exp \left[-i \int_{\gamma(t)} \alpha \right] \psi(\gamma(0)). \quad (35)$$

Consider the case when $\gamma(t)$ is a closed curve: $\gamma(1) = \gamma(0)$. Although $\gamma(1)$ and $\gamma(0)$ are the same point in M^{dN} the wave functions $\psi(\gamma(1))$ and $\psi(\gamma(0))$ differ by the phase factor

$$\chi(\gamma) = \exp \left[-i \oint_{\gamma} \alpha \right], \quad (36)$$

which is an element of the holonomy group of connections. Its value is determined by the de Rham cohomology class to which α belongs and the homology class $\{\gamma\}$ to which γ belongs. For a given α Eq. (36) gives a one-dimensional representation of the homology group $H_1(M^{dN})$. Since

$$H_1(M_{dN}) = \mathbb{Z}_2, \quad \text{for } d \geq 3, \quad (37)$$

and

$$H_1(M^{2N}) = \mathbb{Z}, \quad (38)$$

we have, for $d \geq 3$,

$$\chi(\gamma) = \begin{cases} 1, & \forall \alpha \in \text{trivial class of } H^1(M^{dN}), \\ \pm 1, & \forall \alpha \in \text{nontrivial class of } H^1(M^{dN}), \end{cases} \quad (39)$$

and for $d = 2$,

$$\chi(\gamma) = e^{in_\gamma \Theta}, \quad \Theta \in \mathbb{R}, n_\gamma \in \mathbb{Z}. \quad (40)$$

One recognizes the χ in Eq. (39) to be just the one-dimensional representations of the permutation group S_N : bosonic in the upper case and fermionic for the lower case. In Eq. (40) n_γ characterizes the homology class to which γ belongs and the parameter Θ is determined by the choice of the connection one-form $\alpha \in H^1(M^{dN})$. Different values of Θ implies different statistics.

IV. FRACTIONAL STATISTICS IN THE $d=2$ CASE

We have seen that the connection one-form α on the quantization bundle $\hat{L}$ satisfies the equations

$$\begin{aligned} h^{-1}\omega &= -d\alpha, \text{ on } \Gamma, \\ d\alpha|_{V(M^{dN})} &= 0. \end{aligned} \quad (41)$$

That is, α is generally determined up to a flat connection one-form α_0 and is itself flat when the vertical polarization is chosen. In the latter case only the de Rham cohomology class $H^1(M^{dN})$ to which α belongs matters to the phase factor $\chi(\gamma)$. In what follows we focus on the case $d=2$. Then, locally, α admits the expression

$$-\alpha = -\delta + \sum_{i=1}^N \sum_{a=1}^2 p_{ia} dq_i^a + \frac{\Theta}{\pi} \sum_{i < j}^N d\phi_{ij}(q), \quad (42)$$

where ϕ_{ij} is the azimuthal angle of particle i relative to particle j . The last term on the right hand side of Eq. (42) is α_0 . This expression obviously obeys Eq. (41) and gives the result for the integral in Eq. (36)

$$- \oint_{\gamma} \alpha = \frac{\Theta}{\pi} \sum_{i < j}^N n_{ij} \pi, \quad (43)$$

where $n_{ij} \equiv \oint_{\gamma} \phi_{ij} / \pi$ is the winding number of the j th particle around the i th particle. Thus the factor is

$$\chi(\gamma) = e^{i\Theta \sum_{i < j}^N n_{ij}} = e^{i\Theta n_{\gamma}}. \quad (44)$$

The choice $\Theta=0 \pmod{2\pi}$ gives Bose–Einstein statistics, $\Theta=\pi \pmod{2\pi}$ gives Fermi–Dirac statistics, $\Theta=\text{rational number}$ gives fractional statistics, and any other choice of real number gives Θ statistics.

A nontrivial choice of α_0 may also contribute a statistical factor to the propagator. To see this, consider a system of N indistinguishable particles interacting with a Hamiltonian

$$H = \frac{1}{2m} \sum_{i=1}^N \sum_{a=1}^2 (p_{ia}^2 + V(q_i^a)). \quad (45)$$

Since H does not preserve the vertical polarization, the evolution from $t=0$ to t generated by H brings the Hilbert space from $\mathcal{H}$ to $\mathcal{H}_t$. According to the discussion given in Sec. II, the method outlined in Eqs. (30)–(32) should be used to quantize the system. There is an explicit expression of Eq. (30) (Ref. 6)

$$\mathcal{K}_t(\psi, \psi_t) = \int_{\Gamma} (i/h)^{dN/2} [\det(X_{q_i^a}, X_{q_j^a}^t)] \bar{\psi}(q) \psi_t(q) |\omega|^{2N}. \quad (46)$$

By means of this expression, the quantum operator of the Hamiltonian equation (45) defined by Eq. (32) may be rewritten as

$$O_H \psi = i\hbar \frac{d}{dt} (\phi_H^t \psi) \Big|_{t=0} = i\hbar \frac{d}{dt} \Psi_t \Big|_{t=0}. \quad (47)$$

Using Eq. (6.49) in Ref. 6, for the $n=dN=2N$ case we have that

$$\Psi_t = \int (i\hbar)^{-N} \left[\det(\omega(X_{q_i^a}, \phi_H^t X_{q_j^b})) \right]^{1/2} \exp \left\{ \frac{i}{\hbar} \int_{\gamma(s)} [-\alpha(X_H) - H] \cdot \phi_H^{-s} ds \right\} \psi(q_i \circ \phi_H^{-t}) d^{2N} p, \quad (48)$$

where $\gamma(s)$ are curves mapping the interval $(0, t)$ to M^{dN} . Changing the integration variables p to $q(0)$, we can rewrite Eq. (48) in the form

$$\Psi_t = \int_{M^{2N}} d^{2N} q(0) K(q(0), 0; q, t) \psi(q(0)), \quad (49)$$

where

$$K(q(0), 0; q, t) = (i\hbar)^{-N} \left[\det \omega(X_{q_i^a}, \phi_H^t X_{q_k^b}) \right]^{1/2} \exp \left\{ \frac{i}{\hbar} \int_{\gamma(s)} [-\alpha(X_H) - H] \cdot \phi_H^{-s} ds \right\} \cdot \left[\det \frac{\partial p_i^a}{\partial q(0)_k^b} \right]. \quad (50)$$

Equation (47) implies that the curves in $\mathcal{X}$ given by $t \rightarrow \exp(-itO_H)\psi(q(0))$ and $t \rightarrow \phi_t\psi(q(0))$ are tangent at $t=0$. Therefore, if we divide the interval $[0, t]$ into M equal subintervals, $\exp(-i(t/M)O_H^F)\psi(q_0)$ can be approximated by $\phi_{t/M}^M\psi$ for large M . Hence

$$\begin{aligned} \exp(-itO_H\psi) &= \lim_{M \rightarrow \infty} [v_{t/M} \cdot \phi_{t/M}^{t/M}]^M \psi((0)) \\ &= \lim_{M \rightarrow \infty} [\phi_{t/M}]^M \psi(q(0)) \\ &= \lim_{M \rightarrow \infty} (-i)^{-MN} \int \prod_{r=0}^{M-1} d^{2N} q_r \exp \left\{ -i \sum_{r=0}^{M-1} S(0, q_r; t/M, q_{r+1}) \right\} \psi(q(0)) \\ &\quad \cdot \det \omega(X_{q_i^{r+1}}, \phi_{t/M} X_{q_j^{r+1}}) \det \left(\frac{\partial p_0^{r+1}}{\partial q_r} \right), \end{aligned} \quad (51)$$

which is equivalent to

$$\langle \psi_f | \psi_i \rangle = \langle \psi(q(0)) | \exp(-itO_H) | \psi(q(0)) \rangle, = \int d^{2N} q(0) d^{2N} q Z(0, q(0); t, q), \quad (52)$$

where

$$Z(0, q(0); t, q(t)) = \int \mathcal{D}q(\gamma(s)) \exp \left\{ i\hbar^{-1} \int_{\gamma(s)} (-\alpha(X_H) - H) ds \right\}. \quad (53)$$

Substituting the connection one-form α given in Eq. (42) into Eq. (53) obtains

$$-\alpha(X_H) - H = \sum_{i=1}^N \sum_{a=1}^2 p_{ia} dq_i^a(X_H) - H + \frac{\Theta}{\pi} \sum_{i < j}^N d\phi_{ij}(X_H) = \mathcal{L} + \frac{\Theta}{\pi} \sum_{i < j}^N d\phi_{ij}(X_H), \quad (54)$$

where $\mathcal{L}$ is the classical Lagrangian corresponding to H . Denote by the set of homology classes of γ 's and write for any class ζ

$$\chi(\xi) = \exp i \frac{\Theta}{\pi} \sum_{i < j}^N (\Delta \phi_{ij})_{\xi}, \quad (55)$$

where along any curve γ in the class ξ , the change in ϕ_{ij} ,

$$(\Delta \phi_{ij})_{\xi} \equiv \int_{\gamma(s) \in \xi} d\phi_{ij}(X_H) ds, \quad (56)$$

depends only on ξ . In terms of $\chi(\xi)$, Eq. (53) becomes

$$Z(q(0), 0; q, t) = \sum_{\{\xi\}} \chi(\xi) \int_{\gamma \in \xi} \mathcal{D}q(\gamma) \exp \left[i \int_0^t L(\gamma) \right]. \quad (57)$$

This is just Eq. (1) in Ref. 4.

V. CONCLUSION

We have seen that in the geometric quantization formalism there exist on the quantization bundle different admissible choices of the flat part α_0 of the connection one-form, to which the classical limit is insensitive. Different statistics of the quantum system are due to the inequivalent de Rham cohomology classes $H^1(M^{dN})$ to which α_0 belongs. In case of $d=2$, for which the topology of the configuration space of N indistinguishable particles is nontrivial, fractional statistics is a consequence of the existence of nontrivial de Rham cohomology classes on M^{dN} . Here we have only discussed cases in which the Hilbert space is composed of the sections of the line bundle. The existence of the internal degrees of freedom which give rise to more complicated bundles and Hilbert spaces would of course be extremely interesting.

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