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On the axial gauge: Ward identities and the separation of infrared and ultraviolet singularities by analytic regularization

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It is shown that the method of analytically regulating Yang–Mills theories in the axial gauge preserves gauge invariance. Two- and three-point Ward identities are computed and verified at the one-loop level. The method also permits a convenient and gauge-invariant separation of infrared and ultraviolet singularities in the axial gauge. In the axial gauge the renormalization constants $Z_3 = Z_1 = 1 + 11g^2C_2/(48\pi\epsilon^2)$, leading to a β function which is identical to that computed in the covariant ξ gauges.

I. INTRODUCTION

Recently, an analytic method[1,2] has been shown to be a powerful and elegant means to implement the principal-value prescription[3] for regulating and evaluating Feynman integrals occurring in Yang–Mills theories in the axial gauges.[4] Such gauges are defined by the constraint $n \cdot A^a = 0$, where A^a_μ is the vector gauge field, n_μ is an arbitrary constant vector, and the superscript a is an index for the gauge group. In particular, analytic representations[1,2] have been found for two-point functions in the general axial gauge ($n^2 \neq 0$) (Ref. 4), the light-cone gauge ($n^2 = 0$) (Ref. 5), and the special gauge defined by $p \cdot n = 0$ ($n^2 \neq 0$) (Ref. 2), where p_μ is the external momentum in the two-point functions.

Although there is a widely held belief (possibly caused by using the work "analytic" to describe different and inequivalent regularization methods[6]) that "analytic regularization does not preserve gauge invariance," it will be demonstrated that the contrary is true for the new analytic method. The preservation of gauge invariance depends on the fact that the method preserves such algebraic properties as commutativity and associativity of operations in the Feynman integrals and that in the appropriate limit the new method, for both the infinite and regular parts of a Feynman integral, yields results that are identical to those obtainable from dimensional regularization. For the light-cone gauge, this has already been demonstrated by verifying two- and three-point Ward identities[7] at the one-loop level.[8]

In Sec. II, we extend this study by verifying these identities in the general axial gauge ($n^2 \neq 0$). Here again, we find that the preservation of algebraic properties mentioned earlier is sufficient to guarantee that Ward identities will be upheld, even before the Feynman integrals involved in the identities are evaluated.

The capability of distinguishing infrared from ultraviolet singularities is another one of the appealing properties of analytic regularization. The lack of this capability in dimensional regularization[9] has been the cause of considerable inconvenience in practical calculations using that technique. In Sec. III, using integrals appearing in the three-point Ward identities as examples, we show how the analytic method can easily be employed to separate the two types of singularities. We also show that when these two types of singularities are not distinguished, the cancellation between the two is the direct cause of the vanishing of some tadpoles.[10] In other words, if infrared and ultraviolet singularities are separated, then not all tadpoles vanish.

In Sec. IV we show that the β function, which can be computed from the renormalization constants Z_3 and Z_1, associated with the self-energy and the three-vertex, respectively, are identical in the axial gauge and the covariant ξ gauges to lowest order. Furthermore, in contrast to the ξ gauges, the equality $Z_1 = Z_3$ in the axial gauge allows the β function to be derived directly from the self-energy.

In Sec. V, we compare our results for the axial gauge with those obtained previously for the light-cone gauge.[8] Briefly, other than being ghost-free, the axial gauge shares the properties of the covariant gauges,[11] but does not have the pecularities possessed by the light-cone gauge. On the other hand, computations in the light-cone gauge are much less tedious.

In the following, we briefly review the analytic representation for the "two-point" integrals—integrals with one external momentum—defined by

$$S_{2\omega}(p,n;\kappa,\mu,\nu,s) \equiv \int d^{\,2\omega}q\,[(p-q)^2]^\kappa (q^2)^\mu (q\cdot n)^{2\nu+s}, \tag{1}$$

where $s = 0$ or 1 and ω, κ, μ, and ν are continuous variables. Feynman integrals in four-dimensional Euclidean space (Minkowski space is reached by analytic continuation) that are sometimes divergent and therefore ill defined, correspond to those in (1) when κ, μ, and ν are integers and $\omega = 2$; these form a subset of (1) which we call primal integrals. Methods of analytic regularization[1] were used to find a representation for (1) in terms of a Meijer G function[12]

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$$
\begin{aligned}
S_{2\omega}(p,n;\kappa,\mu,\nu,s) &= \frac{\pi^2 (p^2)^{\omega+\kappa+\mu+\nu}(n^2)^{\nu}(p\cdot n)^s \Gamma(s+\nu+\frac{1}{2})}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)\Gamma(2\omega+\kappa+\mu+2\nu+s)} \\
&\quad\times G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1-\omega-\mu-\nu-s, 1+\omega+\kappa+\mu+\nu, 1+\nu;\\ 0,\omega+\kappa+\nu; 1/2-s\end{matrix}\right), \quad |y|<1, \\
&= \frac{\pi^2 (p^2)^{\omega+\kappa+\mu}(p\cdot n)^{2\nu+s}\Gamma(s+\nu+\frac{1}{2})}{\Gamma(-\kappa)\Gamma(-\mu)\Gamma(-\nu)\Gamma(2\omega+\kappa+\mu+2\nu+s)} \\
&\quad\times G^{3,2}_{3,3}\left(1/y\,\middle|\,\begin{matrix}1+\nu, 1-\omega-\kappa; 1/2+s+\nu\\ 0,-\omega-\kappa-\mu,\omega+\mu+2\nu+s;\end{matrix}\right), \quad |y|>1,
\end{aligned}
\tag{2}
$$

where $y\equiv (p\cdot n)^2/(p^2n^2)$. The right-hand side of (2) is a well-defined, analytic function of all its variables and, when $n^2\neq 0$, has at most simple poles in the (ω,κ,μ,ν) plane.

The evaluation of any primal integral in terms of the independent variables y, p, n and infinitesimals ϵ_i, which label the singularities, now becomes a well-defined mechanistic process, which is discussed elsewhere.[1] Tables of primal integrals have also been prepared.[13]

II. WARD IDENTITIES

We shall verify both the one-loop radiative corrections to the two-point Ward identity

$$p_\mu \Pi^t_{\mu\nu}(p) = \Pi^t_{\nu\mu}(p)p_\mu = 0, \tag{3}$$

as well as a special case of the three-point Ward identity

$$
\begin{aligned}
ip_\lambda \Gamma^{abc}_{\lambda\mu\nu}(p,-p,0) &\equiv if^{abc}p_\lambda \Gamma_{\lambda\mu\nu}(p,-p,0)\\
&= gf^{abc}\Pi^t_{\mu\nu}(p),
\end{aligned}
\tag{4}
$$

where $\Pi^t_{\mu\nu}(p)$ is the self-energy less its $O(g^0)$ longitudinal term

$$\Pi^t_{\mu\nu}(p) = \Pi_{\mu\nu}(p) - (i/\xi)n_\mu n_\nu, \tag{5}$$

g is the Yang–Mills coupling, f^{abc} are the structure constants of the gauge group, and ξ is the parameter of the gauge-fixing Lagrangian $-(1/2\xi)(n\cdot A^a)^2$; the ghost-free axial gauge is realized in the limit $\xi\to 0$. For a derivation of the identities (3) and (4) see Ref. 8.

Equation (3) expresses the well-known notion that $\Pi^t_{\mu\nu}(p)$ is transverse to p_μ. Because the subtracted term in (5) appears only in the zeroth order (in g), (3) also implies that all radiative corrections to $\Pi_{\mu\nu}$ are transverse. The transversility of $\Pi^t_{\mu\nu}$ may be expressed by writing

$$\Pi^t_{\mu\nu}(p) = -i[\Pi_0(p)P_{\mu\nu} + \Pi_1(p)N_{\mu\nu}], \tag{6}$$

where $P_{\mu\nu}$ and $N_{\mu\nu}$ are two linearly independent tensors transverse to p_μ:

$$P_{\mu\nu} \equiv p^2\delta_{\mu\nu} - p_\mu p_\nu, \tag{7a}$$

$$N_{\mu\nu} \equiv p_\mu p_\nu - (p_\mu n_\nu + p_\nu n_\mu)p^2/p\cdot n + n_\mu n_\nu p^4/(p\cdot n)^2. \tag{7b}$$

Our first task is to verify that, to one-loop order (see Fig. 1), $\Pi^t_{\mu\nu}$ indeed has the form (6), and to also compute the coefficients $\Pi^{(1)}_0$ and $\Pi^{(1)}_1$ in the limit $\xi=0$. (The superscript denotes that these are one-loop results.)

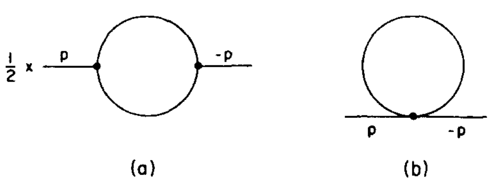

FIG. 1. Diagrams for $O(g^2)$ self-energy. Part (b) is tadpolelike and vanishes only in the limit (10).

The symbol $\Gamma_{\lambda\mu\nu}(p,-p,0)$ in (4) represents the three-vertex with one external line carrying zero momentum; this special momentum configuration reduces the three-vertex to a two-point function. Feynman diagrams for the one-loop radiative corrections to $\Gamma_{\lambda\mu\nu}$ are shown in Fig. 2.

We now digress to explain how it is possible to evaluate the tensor $\Pi^t_{\mu\nu}$, which clearly depends on integrals with nonscalar integrands, using only the scalar integral (1). We parenthetically note that it would have been impossible to derive a viable analytic regularization with generalized exponents, if it were necessary to find representations for integrals with tensorial integrands.

There are four linearly independent, symmetric, rank-two tensors in the axial gauge: $\delta_{\mu\nu}, p_\mu p_\nu, p_\mu n_\nu + p_\nu n_\mu$, and $n_\mu n_\nu$. Therefore any symmetric rank-two tensor such as $\Pi^t_{\mu\nu}$ can be expressed as

$$
\begin{aligned}
\Pi^t_{\mu\nu} = \tfrac{1}{2}[&A_1\delta_{\mu\nu} + A_2 p_\mu p_\nu/p^2\\
&+ A_3(p_\mu n_\nu + p_\nu n_\mu)/p\cdot n\\
&+ p^2 A_4 n_\mu n_\nu/(p\cdot n)^2]/(\zeta - 1),
\end{aligned}
\tag{8}
$$

where A_i are scalar functions of p^2, n^2, and $p\cdot n$, and $\zeta = 1/y$. These functions can be found by contracting $\Pi^t_{\mu\nu}$ as follows:

$$a_1 = \Pi^t_{\mu\nu}\delta_{\mu\nu}, \tag{9a}$$

$$a_2 = \Pi^t_{\mu\nu}p_\mu p_\nu/p^2, \tag{9b}$$

$$a_3 = \Pi^t_{\mu\nu}p_\mu n_\nu/p\cdot n, \tag{9c}$$

$$a_4 = \Pi^t_{\mu\nu}n_\mu n_\nu p^2/(p\cdot n)^2, \tag{9d}$$

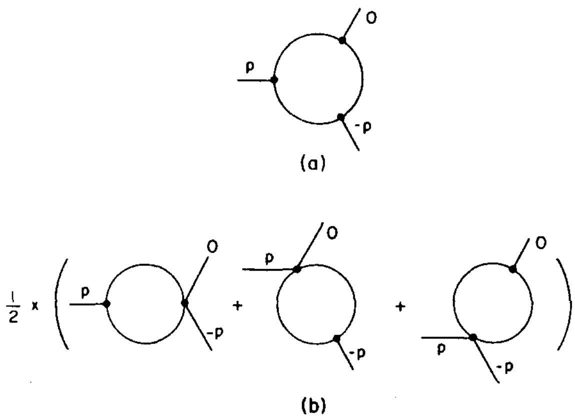

FIG. 2. Diagrams for $O(g^2)$ three-vertex. All momenta flow in. The last diagram in part (b) vanishes identically because of gauge group and Lorentz symmetry.

and solving the resulting set of equations. The relations between a_i and A_i are given in Appendix A. This technique can be generalized for computing tensors of any rank. Since any such tensor can be evaluated by calculating the corresponding scalars a_i, it follows that (for two-point functions) only scalar integrals of the type (1) need ever be computed.

We now return to follow the simpler of two paths. On this path all primal integrals are evaluated in a limit in which the infrared and ultraviolet singularities are not distinguished. (In the next section we consider another limit in which the two types of singularities are distinguished.) For now, we let

$$\kappa = K, \quad \mu = M, \quad \nu = N, \quad \omega = 2 + \epsilon, \tag{10}$$

where ϵ is small but finite. This limit is equivalent[1] to that obtainable with the principal-value prescription used in conjunction with dimensional regularization. We emphasize that with (10) and analytic regularization, all tadpole-like integrals, namely primal integrals satisfying either one or both of the conditions

$$\text{(i)} \quad K \geqslant 0, \tag{11a}$$

$$\text{(ii)} \quad M \geqslant 0, \quad N \geqslant 0, \tag{11b}$$

vanish. In comparison, dimensionless regularization, having only one generalized variable ω, is insufficiently powerful to regulate such integrals. Conventionally,[10] such integrals are simply assumed to be zero[14] in dimensional regularization.

Using the method described above, we find that $\Pi^{\,i}_{\mu\nu}$ indeed has the form of the right-hand side of (6) [this implies that the a_2 and a_3 of (9) vanish, or equivalently, $A_1 + A_2 = A_4 = -A_3$]. The A_1 and A_3 are listed in Table IV of Appendix B in terms of reduced primal integrals defined by

$$\widehat{S}(K,M,N,s) \equiv \pi^{-\omega}(p^2)^{-K-M-2}(p\cdot n)^{-2N-s} \times S_4(p,n;K,M,N,s). \tag{12}$$

Because of space limitations, all integrals satisfying (11) have been omitted from Table IV.

In the limit (10), using (2) we find

$$\Pi_0 = \frac{g^2C_2}{32\pi^2}\frac{1}{1-\zeta}\left[\frac{22}{3\hat{e}}(1-\zeta) - \ln\left(\frac{4}{\zeta}\right)(8-6\zeta+\zeta^2) - \frac{62}{9} + \frac{44\zeta}{9} + 2\zeta^2 + \left(\frac{8}{\zeta} - 8 + 2\zeta - \frac{\zeta^2}{2}\right)Z\right], \tag{13}$$

$$\Pi_1 = \frac{g^2C_2}{32\pi^2}\frac{1}{1-\zeta} \times\left[-\frac{10}{3} + 2\zeta + \ln\left(\frac{4}{\zeta}\right)\left(7-\zeta-\frac{9}{1-\zeta}\right) - \frac{1}{2}\left(\frac{16}{\zeta} - 5 + \zeta - \frac{9}{1-\zeta}\right)Z\right], \tag{14}$$

where

$$1/\hat{e} \equiv 1/\epsilon + \ln p^2 + \gamma, \tag{15}$$

$\zeta = 1/y$, γ is Euler's constant, and Z is defined in Table II.

Noticeably absent from Π_1 is an infinite part. This will allow us to comment in Sec. III on the multiplicative renormalizability of the Yang–Mills theory in the axial gauge after having ascertained that the finiteness of Π_1 is not the result of a cancellation between the as yet undistinguished infrared and ultraviolet singularities.

We now consider the three-point Ward identity. To verify (4), it is sufficient to expand the left-hand side, which is again a symmetric rank-two tensor, as in (9). However, to learn more about the three-vertex $\Gamma_{\mu\nu\lambda}$, we shall instead calculate it explicitly. This vertex function has the symmetry relation

$$\Gamma_{\lambda\mu\nu}(p,-p,0) = -\Gamma_{\mu\lambda\nu}(-p,p,0), \tag{16}$$

showing that it can be expanded in terms of ten independent tensors $T^{(i)}_{\lambda\mu\nu}$ given in Table I:

$$\Gamma_{\lambda\mu\nu}(p,-p,0) = \frac{g}{2(\zeta-1)}\sum_{i=1}^{10} B_i\, T^{(i)}_{\lambda\mu\nu}. \tag{17}$$

In a manner similar to (9), we compute the ten scalar functions B_i by contracting both sides of (17) using the symbol b_i to label the contracted left-hand sides. The linear relations between the sets B_i and b_i are given in Appendix A. The formidable expressions for B_i, in the form of a sum of primal integrals with tadpoles omitted, are given in Table V of Appendix B. When the integrals are evaluated using (2) in the limit (10) the B_i's have values given in Table II.

To satisfy (4), the B_i's must first satisfy the "transversality" relations

$$B_3 + (B_6 + B_7)/(\zeta - 1) = B_4 + (B_5 + B_8)/(\zeta - 1) = -(B_6 + B_{10})/(\zeta - 1) \tag{18a}$$

and

$$B_1 + 2B_2 + B_3 + B_4 = -(\zeta - 1)(B_6 + B_7 + B_8 + B_9), \tag{18b}$$

TABLE I. The ten tensors $T^{\,i}_{\lambda\mu\nu}$ of (17) and the operators $O^{\,i}_{\lambda\mu\nu}$ of (A4).

i	$T^{\,i}_{\lambda\mu\nu}$	$O^{\,i}_{\lambda\mu\nu}$
1	$\delta_{\lambda\mu}\,p_\nu$	$\delta_{\lambda\mu}\,p_\nu/p^2$
2	$\delta_{\mu\nu}\,p_\lambda + \delta_{\lambda\nu}\,p_\mu$	$\delta_{\mu\nu}\,p_\lambda/p^2$
3	$p^2/p\cdot n\delta_{\lambda\mu}\,n_\nu$	$\delta_{\lambda\mu}\,n_\nu/(p\cdot n)$
4	$p^2/p\cdot n(\delta_{\mu\nu}\,n_\lambda + \delta_{\nu\lambda}\,n_\mu)$	$\delta_{\mu\nu}\,n_\lambda/p\cdot n$
5	$p^2/(p\cdot n)^2 n_\lambda n_\mu\,p_\nu/(\zeta-1)$	$n_\lambda n_\mu\,p_\nu/(p\cdot n)^2/(\zeta-1)$
6	$p^2/(p\cdot n)^2(n_\mu n_\nu\,p_\lambda + n_\nu\,n_\lambda\,p_\mu)/(\zeta-1)$	$n_\mu\,n_\nu\,p_\lambda/(p\cdot n)^2/(\zeta-1)$
7	$p_\lambda\,p_\mu\,n_\nu/(p\cdot n)/(\zeta-1)$	$p_\lambda\,p_\mu\,n_\nu/p^2/p\cdot n(\zeta-1)$
8	$(p_\mu\,p_\nu\,n_\lambda + p_\nu\,p_\lambda\,n_\mu)/(p\cdot n)/(\zeta-1)$	$p_\mu\,p_\nu\,n_\lambda/p^2/p\cdot n/(\zeta-1)$
9	$p_\lambda\,p_\mu\,p_\nu/p^2/(\zeta-1)$	$p_\lambda\,p_\mu\,p_\nu/p^4/(\zeta-1)$
10	$p^4/(p\cdot n)^3 n_\lambda n_\mu n_\nu/(\zeta-1)$	$p^2 n_\lambda n_\mu n_\nu/(p\cdot n)^3/(\zeta-1)$

TABLE II. Coefficients (in units of $g^2C_2/32\pi^2$) B_i of (17) after evaluating all the integrals in Table V.[a] Note: $1/\hat{e} = 1/\epsilon + \gamma + \log(p^2)$.

$$B_1 = -\tfrac{88}{3}*1/\hat{e}*(1-1/y) - \tfrac{304}{9} - 20/y^2 + 376/(9y)$$
$$+ \log(4y)*(46 + 10/y^2 - 34/y - 18y/(1-y)) + Z*(25 - 32y + 5/y^2 - 15/y + 9y/(1-y)),$$
$$B_2 = \tfrac{44}{3}*1/\hat{e}*(1 - 1y) - \tfrac{268}{9} + 4/y^2 + 124/(9y)$$
$$+ \log(4y)*(-2 - 2/y^2 + 10/y - 18y/(1-y)) + Z*(-11 - 1/y^2 + 3/y + 9y/(1-y)),$$
$$B_3 = 32 + 12/y^2 - 32/y + \log(4y)*(-14 - 6/y^2 + 10/y + 18y/(1-y)) + z*(7 - 3/y^2 + 7/y - 9y/(1-y)),$$
$$B_4 = 16 - 4/y + \log(4y)*(-14 + 2/y + 18y/(1-y)) + Z*(-5 + 16y + 1/y - 9y/(1-y)),$$
$$B_5 = \tfrac{328}{3} + 20/y^2 - 208/(3y) + \log(4y)*(-70 - 10/y^2 + 50/y + 90y/(1-y))$$
$$+ Z*(-37 + 64y - 5/y^2 + 23/y - 45y/(1-y)),$$
$$B_6 = \tfrac{268}{3} + 8/y^2 - 112/(3y) + \log(4y)*(26 - 4/y^2 + 2/y + 90y/(1-y))$$
$$+ Z*(-1 + 16y - 2/y^2 + 5/y - 45y/(1-y)),$$
$$B_7 = -\tfrac{152}{3} - 12/y^3 + 40/y^2 - 112/(3y) + \log(4y)*(-90 + 6/y^3 - 14/y^2 + 38/y - 90y/(1-y))$$
$$+ Z*(5 + 3/y^3 - 9/y^2 + 1/y + 45y/(1-y)),$$
$$B_8 = -\tfrac{206}{3} - 12/y^2 + 116/(3y) + \log(4y)*(6 + 6/y^2 - 18/y - 90y/(1-y))$$
$$+ Z*(13 - 32y + 3/y^2 - 11/y + 45y/(1-y)),$$
$$B_9 = \tfrac{8}{3} + 12/y^3 - 208/(3y^2) + 344/(3y) + \log(4y)*(90 - 6/y^3 + 14/y^2 - 38/y + 90y/(1-y))$$
$$+ Z*(-5 - 3/y^3 + 9/y^2 - 1/y - 45y/(1-y)),$$
$$B_{10} = -96 - 12/y^2 + 48/y + \log(4y)*(6 + 6/y^2 - 18/y - 90y/(1-y))$$
$$+ Z*(13 - 32y + 3/y^2 - 11/y + 45y/(1-y)).$$

[a] In the above tables, we use the symbol Z to denote the infinite series

$$Z = 2\sum_{l=0}^{\infty} \frac{(1)_l\, y^l}{(3/2)_l}\left[\ln y - \psi(1+l) + \psi\left(\frac{3}{2}+l\right)\right], \quad |y| \leqslant 1$$
$$= \frac{1}{2\sqrt{\pi}}\sum_{l=0}^{\infty} \frac{(\frac{1}{2})_l\, y^{-l-1}}{(1)_l}\left\{\left[\psi\left(\frac{1}{2}-l\right) - \psi(1+l) - \ln y\right]^2 + 2\psi'\left(\frac{1}{2}\right) - \psi'(1+l) - \psi'\left(\frac{1}{2}-l\right)\right\}, \quad |y| > 1$$
$$= \sqrt{\pi}\, G^{2,3}_{3,3}\left(y \,\middle|\, \begin{matrix} 0,0,0; \\ 0,0;\, -\frac{1}{2} \end{matrix}\right), \quad \text{for all } y.$$

which they do. Then they must satisfy

$$B_3 + (B_6 + B_7)/(\zeta - 1) = -A_3, \tag{18c}$$
$$B_2 + B_4 = -A_1, \tag{18d}$$

which they also do, both in terms of primal integrals (from Tables IV and V) and when the intervals are evaluated [from (13), (14), and Table II]. This concludes our demonstration by explicit computation that the two- and three-point Ward identities are satisfied in our method, in the limit (10).

We wish to emphasize a rather appealing feature of the analytic method, that is, its ability to display the equality (4) *before* the primal integrals on both sides of the equations have been evaluated[15] by any means of regularization. In our opinion, it is this feature that truly demonstrates the superior properties of the analytic method: it allows one to manipulate divergent integrals using the same formal rules of algebra as are used for finite integrals.

In Table II, the absence of infinite parts in B_i, $i = 3$–10, is crucial for multiplicative renormalizability. However, a final assessment of this desirable result again must be deferred until after we have separated the infrared and ultraviolet singularities in the primal integrals. This is done in the next section.

III. TADPOLES AND THE SEPARATION OF INFRARED AND ULTRAVIOLET SINGULARITIES

The capability of evaluating tadpole integrals[10] defined by the conditions (11) is special to analytic regularization. To illustrate why dimensional regularization is incapable of regulating a tadpole integral consider a simple case of (11) with $K = N = 0$, and M arbitrary

$$I(\omega) \equiv \int d^{2\omega}q(q^2)^M. \tag{19}$$

Clearly, if $\mathrm{Re}(\omega + M) \geqslant 0$, $I(\omega)$ is ultraviolet divergent and if $\mathrm{Re}(\omega + M) \leqslant 0$, $I(\omega)$ is infrared divergent. So regardless of the value of M, there does not exist a region in the entire complex ω plane in which $I(\omega)$ can be well defined. Therefore, as a function of ω alone, $I(\omega)$ cannot be evaluated by analytic continuation. In other words, the integral cannot be regulated by dimensional regularization.

In our analytic regularization, all two-point integrals (1), of which (19) is but a special case, are considered to be functions over the complex (ω,κ,μ,ν) (hyper) space, not merely as functions of ω. Although no region in the ω plane exists in which $I(\omega)$ is regular, there always exists a region in the (ω,κ,μ,ν) space in which the generalized integral is regular.

Therefore it is always possible to evaluate (1) by analytic continuation. This is why dimension regularization must set the right-hand side of (19) equal to zero (or some other value) by decree, whereas in analytic regularization, the value of any given primal integral with parameters K, M, and N is determined by the limiting processes $(\omega,\kappa,\mu,\nu) \to (\omega,K,M,N) \to (2,K,M,N)$. In the limiting process (10) used in the last section, all tadpole integrals vanish. See Ref. 1 for more details.

We now return to (19) and consider the special case $M = -2$. By power counting it is clear that the integral is both ultraviolet and infrared divergent when ω approaches 2. Why then should it be finite and zero in analytic regularization? The answer is simple: In the special limiting process (10) both the infinite and the finite parts associated, respectively, with the infrared and ultraviolet singularities exactly cancel! (See Table 3 of Ref. 13 for tabulation of this and other integrals.) That these two types of singularities can cancel each other is certainly consistent with the spirit of dimensional regularization because in this method all singularities must be expressed as $1/\epsilon$ poles and are therefore indistinguishable. Similarly, it has already been shown elsewhere[1] that the vanishing of many tadpole integrals defined by (11) is caused by the cancellation between infrared and ultraviolet singularities.[16]

In dimensional regularization, with sufficient knowledge of what the outcome should be, it is possible to separate infrared from ultraviolet singularities.[17] A common practice is to to assign a mass to each massless particle. Other than being cumbersome, this procedure is also very tricky and must be practiced with great care because it does not preserve gauge invariance.

In our analytic method, infrared and ultraviolet singularities can be easily separated by a judicious choice of limiting procedure. The simplest limit that serves this purpose (but does not distinguish the two types of infrared singularities[16]) is

$$\kappa = K + \rho, \quad \mu = M + \rho, \quad \nu = N, \quad \omega = 2 + \epsilon, \quad \rho \to 0, \tag{20}$$

with ϵ small but finite. In this limit, ultraviolet singularities are characterized by the pole

$$1/\hat{e}_1 = -1/\epsilon_1 + \ln p^2 + \gamma, \quad \epsilon_1 = -2\rho - \epsilon, \tag{21a}$$

and both infrared singularities by

$$1/\hat{e}_0 = 1/\epsilon_0 + \ln p^2 + \gamma, \quad \epsilon_0 = \epsilon + \rho. \tag{21b}$$

Tadpole integrals that are both ultraviolet and infrared divergent are proportional to $1/\hat{e}_0 - 1/\hat{e}_1$. The limit (10) is a special case of (20) with $\rho = 0$, for in that limit such integrals vanish:

$$(1/\hat{e}_0 - 1/\hat{e}_1)_{\rho=0} = 1/\epsilon - 1/\epsilon = 0. \tag{22}$$

We now use limit (20) to evaluate the integrals in Tables IV and V. In this limit tadpole diagrams such as (b) of Fig. 1 contain terms that do not automatically vanish. However, the ultraviolet and infrared singularities of the regulated integrals describing Fig. (1b) cancel among themselves, so that in the limit (20) this diagram does vanish. Remarkably, the results for the remaining diagrams are identical to those in (13), (14), and Table II except that all pole terms $(1/\hat{e})$ therein are now replaced by ultraviolet poles $(1/\hat{e}_1)$; all infrared singularities have cancelled! We do not know whether this is a general result [i.e., whether we can use the simpler limit (10) in which all tadpole integrals vanish and treat all poles as ultraviolet singularities] or if not, what is the reason for this remarkable cancellation of infrared singularities at the one-loop level. We do note, however, that the Ward identities (18) do not appear to be manifestly satisfied prior to regularization, if all tadpole integrals and diagrams are retained.

IV. THE β FUNCTION

Our calculation shows that at the one-loop level, all infinite parts in $\Pi_{\mu\nu}$ and $\Gamma_{\lambda\mu\nu}$ are of ultraviolet origin, and of the operators generated by radiative correction, only those that appear in the original Lagrangian at the tree (no-loop) level—$\Pi_0 P_{\mu\nu}$ in $\Pi_{\mu\nu}$ and $B_i O^{(i)}_{\lambda\mu\nu}$, $i = 1,2,3$ in $\Gamma_{\lambda\mu\nu}$—have infinite parts. These results indicate that Yang–Mills theories are multiplicatively renormalizable in the axial gauge. That is, the infinite parts generated by radiative corrections can be absorbed into renormalization constants Z_1 and Z_3 that rescale the gauge field, the coupling constant, and the gauge parameter according to

$$A^a_\mu \to \bar{A}^a_\mu = Z_3^{-1/2} A^a_\mu, \tag{23a}$$

$$g \to \bar{g} = Z_1 Z_3^{-3/2} g \equiv Z_g\, g, \tag{23b}$$

so that the bare Lagrangian is

$$\mathscr{L}(\bar{A},\bar{g},\bar{\xi}) = -\tfrac{1}{4}[F^a_{\mu\nu}(\bar{A},\bar{g})]^2 - (1/2\bar{\xi})(n \cdot \bar{A}^a_\mu)^2, \tag{24}$$

and from (13) and Table II

$$Z_3^{\text{axial}} = Z_1^{\text{axial}} = 1 + \frac{g^2 C_2}{16\pi^2}\frac{11}{3}\left[\frac{1}{\epsilon} + \ln\left(\frac{p^2}{\lambda^2}\right) + \cdots\right], \tag{25}$$

where the explicit dependence on an arbitrary momentum scale λ is displayed. As expected, the two renormalization constants are identical in the axial gauge, an outcome which is not true in the covariant ξ gauges, where[18]

$$Z_1^\xi - 1 = \frac{g^2 C_2}{32\pi^2}\left(\frac{17}{6} - \frac{3}{2}\xi\right)\left[\frac{1}{\epsilon} + \cdots\right],$$

$$Z_3^\xi - 1 = \frac{g^2 C_2}{32\pi^2}\left(\frac{13}{3} - \xi\right)\left[\frac{1}{\epsilon} + \cdots\right]. \tag{26}$$

Although the renormalization constants are gauge dependent, the β function, or the logarithmic derivative of the renormalized coupling constant,

$$\beta(g) \equiv \lambda\frac{\partial g}{\partial\lambda} = 2gZ_g^{-1}\frac{\partial Z_g}{\partial(1/\epsilon)}, \tag{27}$$

should be gauge independent. This is indeed true since

$$Z_g^{\text{axial}} = (Z_3^{\text{axial}})^{-1/2} = Z_g^\xi = 1 - \frac{g^2 C_2}{32\pi^2}\frac{11}{3\epsilon} + O(g^4), \tag{28}$$

which leads to the equality

$$\beta(g)^{\text{axial}} = \beta(g)^\xi = -\frac{g^3 C_2}{16\pi^2}\frac{11}{3\epsilon} + O(g^5). \tag{29}$$

The point to be noted here is that unlike the ξ gauges, in the axial gauge the equality (25) allows the β function to be derived directly from radiative corrections to the self-energy.

V. AXIAL GAUGE VERSUS LIGHT-CONE GAUGE

A comparison of our result for the axial gauge with results obtained previously[2,8] for the principal-value prescription of the light-cone gauge ($n^2 = 0$) is summarized in Table III, with the following comments.

(i) The advantages for the light-cone gauge is the simplicity of the propagator and the extreme ease with which Feynman integrals can be evaluated. On the other hand, we emphasize that although integrals in the axial gauge are comparatively more cumbersome to compute, with the aid of analytic regularization, such computations are not the kind of brutal undertaking they used to be when the principal-value prescription was used. In any case, two-point integrals in both the axial and light-cone gauges have now been evaluated and tabulated.[13]

(ii) In this paper and in Ref. 8, analytic regularization has been employed to verify that two- and three-point Ward identities in Yang–Mills theories are true in both gauges.

(iii) A pecularity of the light-cone gauge is that some of the divergences generated by one-loop radiative corrections manifest themselves[8,19] as double poles $[O(1/\epsilon^2)]$. This effect is directly caused by the coalescence of ultraviolet divergences with one type of infrared divergence inherent in the analytic regularization of this gauge; only one other Lorentz invariant regularization of this gauge exists,[20] which manifests a nonlocal infinite part residual to the double pole. In this aspect the axial gauge is normal: one-loop corrections generate only single poles and local interactions.

(iv) We have shown that in the axial gauge infrared and ultraviolet singularities can be separated by letting the generalized integrals approach a given primal integral in an appropriate way [see (20)]. In contrast, for the same reason given in (iii) these singularities cannot be separated in the light-cone gauge. Indeed, when the limit (20) is used to evaluate the integrals in the three-point Ward identity, we find that the identity is no longer true. The only limit that we believe does not lead to any incorrect result in the light-cone gauge is (10), in which all tadpole-like integrals (defined by $K \geqslant 0$ and/or $M \geqslant 0$ in this gauge) vanish.

(v) In the axial gauge, infinite parts generated by one-loop corrections occur only in operators associated with the original Lagrangian at the tree (zero-loop) level. Therefore, as is well-known, the theory in this gauge is multiplicatively renormalizable. In contrast, in the light-cone gauge, new operators generated by radiative corrections also have infinite parts.[8,19,21] Consequently, the theory in this gauge, assuming it is renormalizable, is not multiplicatively renormalizable. The renormalization program in the light-cone gauge needs to be thoroughly studied.

Note added in proof: All comments in this paper pertaining to the peculiarities of the light-cone gauge refer to the principal-value prescription of that gauge. Recent calculations by the authors (Chalk River preprint CRNL-TP-85-II-11) have shown that the Mandelstam–Leibbrandt prescription of the gauge does not share such peculiarities; in particular it is one-loop renormalizable.

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APPENDIX A: CALCULATION OF SCALAR FUNCTIONS A_i AND B_i

Define the scalar functions A_i by the general expansion for the self-energy (8) and compute the scalar quantities a_i defined in (9). Substituting (8) into (9) yields the linear relations

$$a_i = (U_\Pi^{-1})_{ij} A_j, \tag{A1}$$

which have inverse relations

$$A_i = (U_\Pi)_{ij}\, a_j. \tag{A2}$$

For general axial gauges, defining $\zeta = p^2 n^2/(p\cdot n)^2 = 1/y$, we find

$$U_\Pi = \frac{1}{\zeta - 1}\begin{bmatrix} (\zeta-1)^2 & -\zeta(\zeta-1) & 2(\zeta-1) & -(\zeta-1) \\ -\zeta(\zeta-1) & 3\zeta^2 & -6\zeta & \zeta+2 \\ \zeta-1 & -3\zeta & 2(\zeta+2) & -3 \\ -(\zeta-1) & \zeta+2 & -6 & 3 \end{bmatrix}, \tag{A3}$$

TABLE III. Comparison of axial and light-cone gauges in analytic regularization.

	Axial gauge	Light-cone gauge (principal-value prescription)
Evaluation of integrals	moderately easy	extremely easy
Preserves gauge invariance	yes	yes
Divergences at one loop	single poles	single and double poles
Infrared and ultraviolet singularities separable	yes [see (20)]	no
"New" operators in $\Pi_{\mu\nu}(p)$ at one loop contain infinite parts	no	yes
"New" operators in $\Gamma_{\lambda\mu\nu}(p,-p,0)$ at one loop contain infinite parts	no	yes
Multiplicatively renormalizable	yes	no (see note added in proof)
Renormalization constants	$Z_1 = Z_3 = 1 + \frac{g^2 C_2}{16\pi^2}\cdot\frac{11}{3\epsilon}$	?
β function	$-\frac{g^2 C_2}{16\pi^2}\cdot\frac{11}{3}$	?

independently of the regularization method.

Similarly, for the vertex function written in (17), we compute the scalar quantities b_i defined by contracting

$$b_i = \sum_{\lambda,\mu,\nu} \Gamma_{\lambda\mu\nu}(p, -p, 0) O^{(i)}_{\lambda\mu\nu}, \tag{A4}$$

where $O^{i}_{\lambda\mu\nu}$ are operators defined in Table I. This defines a linear relation between B_i and b_i which, upon inversion, yields

$$B_i = (U_\Gamma)_{ij}\, b_j. \tag{A5}$$

We find

$$U_\Gamma = \frac{1}{2(\zeta - 1)} \times \begin{bmatrix} \zeta & 0 & -1 & 0 & -\zeta & -2 & \zeta & 2\zeta & -\zeta^2 & 1 \\ 0 & \zeta & 0 & -1 & -1 & -\zeta-1 & \zeta & 2\zeta & -\zeta^2 & 1 \\ -1 & 0 & 1 & 0 & 1 & 2 & -\zeta & -2 & \zeta & -1 \\ 0 & -1 & 0 & 1 & 1 & 2 & -1 & -\zeta-1 & \zeta & -1 \\ -\zeta & -2 & 1 & 2 & 3\zeta+2 & 10 & -\zeta-4 & -8\zeta-2 & \zeta(\zeta+4) & -5 \\ -1 & -\zeta-1 & 1 & 2 & 5 & 3\zeta+7 & -4\zeta-1 & -5\zeta-5 & \zeta(\zeta+4) & -5 \\ \zeta & 2\zeta & -\zeta & -2 & -\zeta-4 & -8\zeta-2 & \zeta(3\zeta+2) & 10\zeta & -5\zeta^2 & \zeta+4 \\ \zeta & 2\zeta & -1 & -\zeta-1 & -4\zeta-1 & -5\zeta-5 & 5\zeta & \zeta(3\zeta+7) & -5\zeta^2 & \zeta+4 \\ -\zeta^2 & -2\zeta^2 & \zeta & 2\zeta & \zeta(\zeta+4) & 2\zeta(\zeta+4) & -5\zeta^2 & -10\zeta^2 & 5\zeta^3 & -3\zeta-2 \\ 1 & 2 & -1 & -2 & -5 & -10 & \zeta+4 & 2\zeta+8 & -3\zeta-2 & 5 \end{bmatrix} \tag{A6}$$

again independently of the method of regularization.

APPENDIX B: REDUCTION TO PRIMAL INTEGRALS

In Tables IV and V we list the one-loop scalar functions A_i and B_i, defined, respectively, in (8) and (17) in terms of reduced primal integrals defined in (12). The A_i's and B_i's are given in units of $g^2C_2/16\pi^2$.

The tables were generated by evaluating the diagrams of Fig. 1, contracting as in (9) and (A4), and using the matrices (A3) and (A6). All contractions were simplified by reducing them to a sum of primal integrals using both the "shift" rule $q \to p - q$ in (1) where necessary, and partial fraction decomposition of integrals with multiple denominators. See Appendix C of Ref. 8 for more details.

Once the coefficients were reduced to a sum of primal integrals, the integrals themselves were reduced to a smaller set by using algebraic identities easily obtainable for cases with $N > 0$. For example,

$$S(K,M,1,0) = (p \cdot n)^2 S(M,K,0,0) - 2p \cdot n S(M,K,0,1) + S(M,K,1,0). \tag{B1}$$

We emphasize that the legitimacy of this technique is founded on the fact that divergent integrals obey the usual rules of algebra, such as (B1). We also note that when regulating primal integrals, it is vital to preserve this property. This is true of both (1) and (20), but it is easy to invent limiting processes which do not preserve simple algebraic identities such as (B1).

Each of the primal integrals was then evaluated according to (2) by an algorithm described elsewhere.[13] Both limits (10) and (20) were investigated. To reduce the tables to manageable proportions, those integrals satisfying (11) are omitted here.

All calculations were performed with the algebraic manipulator SCHOONSCHIP,[22] except for the matrices (A3) and (A6) that were obtained using REDUCE.[23] The tables themselves were formatted, using an on-line editor and typewriter, directly from the computer output.

TABLE IV. Coefficients A_i of (8) in terms of primal integrals with tadpoles omitted.

$$
\begin{aligned}
A_1 = {} & -(-1/y^3 + 2/y^2)\ast\widehat{S}(-1,0,-1,0) - (-2/y^3 + 6/y^2 - 8/y)\ast\widehat{S}(-1,0,-1,1)\\
& - (1/(2y^3))\ast\widehat{S}(-1,1,-1,0) - (1/y^3 - 2/y^2)\ast\widehat{S}(-1,1,-1,1)\\
& - (12 - 10/y)\ast\widehat{S}(-1,-1,0,0) - 8\ast\widehat{S}(-1,-1,1,0) - (1/(2y^3) - 4/y^2 + 4/y)\ast\widehat{S}(-1,-1,-1,0)\\
& - (-16 + 1/y^3 - 4/y^2 + 16/y)\ast\widehat{S}(-1,-1,-1,1),\\
A_2 = {} & + (3 - 1/y^3 + 1/y^2 - 1/y + 3y/(1-y))\ast\widehat{S}(-1,0,-1,0)\\
& + (-12 - 2/y^3 + 4/y^2 - 12y/(1-y))\ast\widehat{S}(-1,0,-1,1)\\
& + (\tfrac{3}{2} + 1/(2y^3) + 1/(2y^2) + 3/(2y) + 3y/(2(1-y)))\ast\widehat{S}(-1,1,-1,0)\\
& + (-3 + 1/y^3 - 1/y^2 - 3/y - 3y/(1-y))\ast\widehat{S}(-1,1,-1,1)\\
& + (10 - 10/y + 6y/(1-y))\ast\widehat{S}(-1,-1,0,0) + (8 + 24y/(1-y))\ast\widehat{S}(-1,-1,1,0)\\
& + (\tfrac{3}{2} + 1/(2y^3) - 7/(2y^2) + 3/(2y) + 3y/(2(1-y)))\ast\widehat{S}(-1,-1,-1,0)\\
& + (-9 + 1/y^3 - 3/y^2 + 11/y - 9y/(1-y))\ast\widehat{S}(-1,-1,-1,1),\\
A_4 = {} & -A_3 = + (3 - 1/y^2 - 1/y + 3y(1-y))\ast\widehat{S}(-1,0,-1,0)\\
& + (-12 - 2/y^2 + 8/y - 12y/(1-y))\ast\widehat{S}(-1,0,-1,1)\\
& + (\tfrac{3}{2} + 1/(2y^2) + 3/(2y) + 3y/(2(1-y)))\ast\widehat{S}(-1,1,-1,0)\\
& + (-3 + 1/y^2 - 3/y - 3y(1-y))\ast\widehat{S}(-1,1,-1,1)\\
& + (-2 + 6y/(1-y))\ast\widehat{S}(-1,-1,0,0)\\
& + (24y/(1-y))\ast\widehat{S}(-1,-1,1,0)\\
& + (\tfrac{3}{2} + 1/(2y^2) - 5/(2y) + 3y/(2(1-y)))\ast\widehat{S}(-1,-1,-1,0)\\
& + (7 + 1/y^2 - 5/y - 9y/(1-y))\ast\widehat{S}(-1,-1,-1,1).
\end{aligned}
$$

TABLE V. Coefficients B_i of (17) in terms of primal integrals with tadpoles omitted.

$$
\begin{aligned}
B_1 = {} & (\tfrac{3}{4} + 3/(4y^3) - 13/(4y^2) + 3/(4y) + 3y/(4(1-y)))\ast\widehat{S}(-2,0,-1,0)\\
& + (-\tfrac{9}{2} + 3/(2y^3) - 9/(2y^2) + 23/(2y) - 9y/(2(1-y)))\ast\widehat{S}(-2,0,-1,1)\\
& + (\tfrac{3}{4} - 3/(4y^3) + 3/(4y^2) + 3/(4y) + 3y/(4(1-y)))\ast\widehat{S}(-2,1,-1,0)\\
& + (-3 - 3/(2y^3) + 3/y^2 - 3/y - 3y/(1-y))\ast\widehat{S}(-2,1,-1,1)\\
& + (\tfrac{1}{4} + 1/(4y^3) + /(4y^2) + 1/(4y) + y/(4(1-y)))\ast\widehat{S}(-2,2,-1,0)\\
& + (-\tfrac{1}{2} + 1/(2y^3) - 1/(2y^2) - 1/(2y) - y/(2(1-y)))\ast\widehat{S}(-2,2,-1,1)\\
& + (\tfrac{1}{4} - 1/(4y^3) + 9(4y^2) - 15/(4y) + y/(4(1-y)))\ast\widehat{S}(-2,-1,-1,0)\\
& + (14 - 1/(2y^3) + 2/y^2 - 6/y - 2y/(1-y))\ast\widehat{S}(-2,-1,-1,1)\\
& + (-\tfrac{9}{4} - 13/(4y^3) + 23/(4y^2) - 9/(4y) - 9y/(4(1-y)))\ast\widehat{S}(-1,0,-1,0)\\
& + (9 - 13/(2y^3) + 13/y^2 - 7/y + 9y/(1-y))\ast\widehat{S}(-1,0,-1,1)\\
& + (-1 + 1/y^3 - 1/y^2 - 1/y - y/(1-y))\ast\widehat{S}(-1,1,-1,0)\\
& + (2 + 2/y^3 - 2/y^2 + 2/y + 2y/(1-y))\ast\widehat{S}(-1,1,-1,1)\\
& + (-6 + 10/y - 6y/(1-y))\ast\widehat{S}(-1,-2,0,0)\\
& + (-12 + 20y/(1-y))\ast\widehat{S}(-1,-2,0,1) + (-8 - 24y/(1-y))\ast\widehat{S}(-1,-2,1,0)\\
& + (16y/(1-y))\ast\widehat{S}(-1,-2,1,1)\\
& + (-\tfrac{1}{4} - 1/(4y^3) + 7/(4y^2) - 1/(4y) - y/(4(1-y)))\ast\widehat{S}(-1,-2,-1,0)\\
& + (2 - 1/(2y^3) + 2/y^2 - 10/y + 2y/(1-y))\ast\widehat{S}(-1,-2,-1,1)\\
& + (18 - 10/y - 6y/(1-y))\ast\widehat{S}(-1,-1,0,0) + (-8 - 24y/(1-y))\ast\widehat{S}(-1,-1,1,0)\\
& + (-\tfrac{3}{2} + 5/(2y^3) - 26/(2y^2) + 29/(2y) - 3y/(2(1-y)))\ast\widehat{S}(-1,-1,-1,0)\\
& + (-23 + 5/y^3 - 15/y^2 + 25/y + 9y/(1-y))\ast\widehat{S}(-1,-1,-1,1),\\
B_2 = {} & (\tfrac{3}{4} + 3/(4y^2) - 5/(4y) + 3y/(4(1-y)))\ast\widehat{S}(-2,0,-1,0)\\
& + (-\tfrac{1}{2} - 1/(2y) - 9y/(2(1-y)))\ast\widehat{S}(2,0,-1,1)
\end{aligned}
$$

TABLE V. (*Continued.*)

$$+ (\tfrac{3}{4} - 3/(4y^2) + 3/(4y) + 3y/(4(1-y)))*\hat{S}(-2,1,-1,0)$$
$$+ (-3 - 3y/(1-y))*\hat{S}(-2,1,-1,1)$$
$$+ (\tfrac{1}{4} + 1/(4y^2) + 1/(4y) + y/(4(1-y)))*\hat{S}(-2,2,-1,0)$$
$$+ (-\tfrac{1}{2} - 1/(2y) - y/(2(1-y)))*\hat{S}(-2,2,-1,1)$$
$$+ (\tfrac{1}{4} - 1/(4y^2) + 1/(4y) + y/(4(1-y)))*\hat{S}(-2,-1,-1,0)$$
$$+ (-2 + 1/y - 2y/(1-y))*\hat{S}(-2,-1,-1,1)$$
$$+ (-\tfrac{9}{4} + 1/y^3 - 7/(4y^2) + 7/(4y) - 9y/(4(1-y)))*\hat{S}(-1,0,-1,0)$$
$$+ (9 + 2/y^3 - 4/y^2 + 9y/(1-y))*\hat{S}(-1,0,-1,1)$$
$$+ (-1 - 1/(2y^3) - 1/y - y/(1-y))*\hat{S}(-1,1,-1,0)$$
$$+ (2 - 1/y^3 + 1/y^2 + 2/y + 2y/(1-y))*\hat{S}(-1,1,-1,1)$$
$$+ (-2 - 6y/(1-y))*\hat{S}(-1,-2,0,0) + (4 + 20y/(1-y))*\hat{S}(-1,-2,0,1)$$
$$+ (-24y/(1-y))*\hat{S}(-1,-2,1,0) + (16y/(1-y))*\hat{S}(-1,-2,1,1)$$
$$+ (-\tfrac{1}{4} + 1/(4y^2) - 1/(4y) - y/(4(1-y)))*\hat{S}(-1,-2,-1,0)$$
$$+ (2 - 1/y + 2y/(1-y))*\hat{S}(-1,-2,-1,1)$$
$$+ (-10 + 10/y - 6y/(1-y))*\hat{S}(-1,-1,0,0)$$
$$+ (-8 - 24y/(1-y))*\hat{S}(-1,-1,1,0)$$
$$+ (-\tfrac{3}{2} - 1/(2y^3) + 7/(2y^2) - 3/(2y) - 3y/(2(1-y)))*\hat{S}(-1,-1,-1,0)$$
$$+ (9 - 1/y^3 + 3/y^2 - 11/y + 9y/(1-y))*\hat{S}(-1,-1,-1,1),$$
$$B_3 = (-\tfrac{3}{4} + 1/(4y^2) + 5/(4y) - 3y/(4(1-y)))*\hat{S}(-2,0,-1,0)$$
$$+ (-\tfrac{7}{2} - 1/(2y^2) + 1/(2y) + 9y/(2(1-y)))*\hat{S}(-2,0,-1,1)$$
$$+ (-\tfrac{3}{4} + 1/(4y^2) - 3/(4y) - 3y/(4(1-y)))*\hat{S}(-2,1,-1,0)$$
$$+ (3 + 1/y^2 - 1/y + 3y/(1-y))*\hat{S}(-2,1,-1,1)$$
$$+ (-\tfrac{1}{4} - 1/(4y^2) - 1/(4y) - y/(4(1-y)))*\hat{S}(-2,2,-1,0)$$
$$+ (\tfrac{1}{2} - 1/(2y^2) + 1/(2y) + y/(2(1-y))*\hat{S}(-2,2,-1,1)$$
$$+ (-\tfrac{1}{4} - 1/(4y^2) + 7/(4y) - y/(4(1-y))*\hat{S}(-2,-1,-1,0)$$
$$+ (-6 - 2/y + 2y/(2-y))*\hat{S}(-2,-1,-1,1) + (1/y^3 - 2/y^2)*\hat{S}(-1,0,-2,1)$$
$$+ (\tfrac{9}{4} + 3/y^3 - 23/(4y^2) + 25/(4y) + 9y/(4(1-y)))*\hat{S}(-1,0,-1,0)$$
$$+ (-9 + 6/y^3 - 11/y^2 + 3/y - 9y/(1-y))*\hat{S}(-1,0,-1,1)$$
$$+ (-1/(2y^3))*\hat{S}(-1,1,-2,1)$$
$$+ (1 - 3/(2y^3) + 2/y^2 + 1/y + y/(1-y))*\hat{S}(-1,1,-1,0)$$
$$+ (-2 - 3/y^3 + 4/y^2 - 2/y - 2y/(1-y))*\hat{S}(-1,1,-1,1)$$
$$+ (-10 + 6y/(1-y))*\hat{S}(-1,-2,0,0) + (20 - 20y/(1-y))*\hat{S}(-1,-2,0,1)$$
$$+ (24y/(1-y))*\hat{S}(-1,-2,1,0) + (-16y/(1-y))*\hat{S}(-1,-2,1,1)$$
$$+ (\tfrac{1}{4} + 1/(4y^2) - 7/(4y) + y/(4(1-y)))*\hat{S}(-1,-2,-1,0)$$
$$+ (6 + 2/y - 2y/(1-y))*\hat{S}(-1,-2,-1,1) + (-2 + 6y/(1-y))*\hat{S}(-1,-1,0,0)$$
$$+ (24y(1-y))*\hat{S} - 1,-1,1,0) + (-1/(2y^3) + 4/y^2 - 4/y)*\hat{S}(-1,-1,-2,1)$$
$$+ (\tfrac{19}{2} - 3/(2y^3) + 7/(2y^2) - 21/(2y) + 3y/(2(1-y)))*\hat{S}(-1,-1,-1,0)$$
$$+ (-9 - 3/y^3 + 7/y^2 + 7/y - 9y/(1-y))*\hat{S}(-1,-1,-1,1),$$
$$B_5 = (\tfrac{15}{4} - 3/(4y^3) + 1/(4y^2) + 17/(4y) - 15y/(4(1-y)))*\hat{S}(-2,0,-1,0)$$
$$+ (\tfrac{29}{2} - 3/(2y^3) + 15/(2y^2) - 35/(2y) + 45y/(2(1-y)))*\hat{S}(-2,0,-1,1)$$
$$+ (-\tfrac{15}{4} + 3/(4y^3) + 9/(4y^2) - 15/(4y) - 15y/(4(1-y)))*\hat{S}(-2,1,-1,0)$$
$$+ (15 + 3/(2y^3) - 6/y^2 + 9/y + 15y/(1-y))*\hat{S}(2,1,-1,1)$$
$$+ (-\tfrac{5}{4} - 1/(4y^3) - 5/(4y^2) - 5/(4y) - 5y/(4(1-y)))*\hat{S}(-2,2,-1,0)$$
$$+ (\tfrac{5}{2} - 1/(2y^3) + 3/(2y^2) + 5/(2y) + 5y/(2(1-y)))*\hat{S}(-2,2,-1,1)$$

$$+\left(-\tfrac{5}{4}+1/(4y^3)-5/(4y^2)+11/(4y)-5y/(4(1-y))\right)\ast\widehat{S}(-2,-1,-1,0)$$
$$+\left(-6+1/(2y^3)-3/y^2+4/y+10y/(1-y)\right)\ast\widehat{S}(-2,-1,-1,1)$$
$$+\left(\tfrac{45}{4}+13/(4y^3)-15/(4y^2)-19/(4y)+45y/(4(1-y))\right)\ast\widehat{S}(-1,0,-1,0)$$
$$+\left(-45+13/(2y^3)-22/y^2+29/y-45y/(1-y)\right)\ast\widehat{S}(-1,0,-1,1)$$
$$+\left(5-1/y^3+1/y^2+5/y+5y/(1-y)\right)\ast\widehat{S}(-1,1,-1,0)$$
$$+\left(-10-2/y^3+4/y^2-10/y-10y/(1-y)\right)\ast\widehat{S}(-1,1,-1,1)$$
$$+\left(22-2/y+30y/(1-y)\right)\ast\widehat{S}(-1,-2,0,0)$$
$$+\left(-28-100y/(1-y)\right)\ast\widehat{S}(-1,-2,0,1)+\left(24+120y/(1-y)\right)\ast\widehat{S}(-1,-2,1,0)$$
$$+\left(-80y/(1-y)\right)\ast\widehat{S}(-1,-2,1,1)$$
$$+\left(\tfrac{5}{4}+1/(4y^3)-7/(4y^2)+5/(4y)+5y/(4(1-y))\right)\ast\widehat{S}(-1,-2,-1,0)$$
$$+\left(-10+1/(2y^3)-3/y^2+8/y-10y/(1-y)\right)\ast\widehat{S}(-1,-2,-1,1)$$
$$+\left(-2+2/y+30y/(1-y)\right)\ast\widehat{S}(-1,-1,0,0)+\left(24+120y/(1-y)\right)\ast\widehat{S}(-1,-1,1,0)$$
$$+\left(\tfrac{15}{2}-5/(2y^3)+23/(2y^2)-33/(2y)+15y/(2(1-y))\right)\ast\widehat{S}(-1,-1,-1,0)$$
$$+\left(19-5/y^3+23/y^2-37/y-45y/(1-y)\right)\ast\widehat{S}(-1,-1,-1,1),$$
$$B_6=\left(-\tfrac{15}{4}-5/(4y^2)+17/(4y)-15y/(4(1-y))\right)\ast\widehat{S}(-2,0,-1,0)$$
$$+\left(\tfrac{13}{2}-1/(2y^2)-3/(2y)+45y/(2(1-y))\right)\ast\widehat{S}(-2,0,-1,1)$$
$$+\left(-\tfrac{15}{4}+7/(4y^2)-7/(4y)-15y/(4(1-y))\right)\ast\widehat{S}(-2,1,-1,0)$$
$$+\left(15+1/y^2-1/y+15y/(1-y)\right)\ast\widehat{S}(-2,1,-1,1)$$
$$+\left(-\tfrac{5}{4}-3/(4y^2)-5/(4y)-5y/(4(1-y))\right)\ast\widehat{S}(-2,2,-1,0)$$
$$+\left(\tfrac{5}{2}-1/(2y^2)+5/(2y)+5y/(2(1-y))\right)\ast\widehat{S}(-2,2,-1,1)$$
$$+\left(-\tfrac{5}{4}+1/(4y^2)+3/(4y)-5y/(4(1-y))\right)\ast\widehat{S}(-2,-1,-1,0)$$
$$+\left(2-2/y+10y/(1-y)\right)\ast\widehat{S}(-2,-1,-1,1)+\left(1/y^3-4/y\right)\ast\widehat{S}(-1,0,-2,1)$$
$$+\left(\tfrac{45}{4}+2/y^3-33/(4y^2)+61/(4y)+45y/(4(1-y))\right)\ast\widehat{S}(-1,0,-1,0)$$
$$+\left(-45+4/y^3-9/y^2-1/y-45y/(1-y)\right)\ast\widehat{S}(-1,0,-1,1)$$
$$+\left(-1/(2y^3)-1/y^2\right)\ast\widehat{S}(-1,1,-2,1)$$
$$+\left(5-1/y^3+4/y^2+5/y+5y/(1-y)\right)\ast\widehat{S}(-1,1,-1,0)$$
$$+\left(-10-2/y^3+4/y^2-10/y-10y/(1-y)\right)\ast\widehat{S}(-1,1,-1,1)$$
$$+\left(2+30y/(1-y)\right)\ast\widehat{S}(-1,-2,0,0)+\left(-4-100y/(1-y)\right)\ast\widehat{S}(-1,-2,0,1)$$
$$+\left(120y/(1-y)\right)\ast\widehat{S}(-1,-2,1,0)+\left(-80y/(1-y)\right)\ast\widehat{S}(-1,-2,1,1)$$
$$+\left(\tfrac{5}{4}-1/(4y^2)-3/(4y)+5y/(4(1-y))\right)\ast\widehat{S}(-1,-2,-1,0)$$
$$+\left(-2+2/y-10y/(1-y)\right)\ast\widehat{S}(-1,-2,-1,1)+\left(26-2/y+30y/(1-y)\right)\ast\widehat{S}(-1,-1,0,0)$$
$$+\left(24+120y/(1-y)\right)\ast\widehat{S}(-1,-1,1,0)+\left(-1/(2y^3)+3/y^2-4/y\right)\ast\widehat{S}(-1,-1,-2,1)$$
$$+\left(\tfrac{31}{2}-1/y^3+5/(2y^2)-13/(2y)+15y/(2(1-y))\right)\ast\widehat{S}(-1,-1,-1,0)$$
$$+\left(-29-2/y^3+5/y^2-1/y-45y/(1-y)\right)\ast\widehat{S}(-1,-1,-1,1),$$
$$B_7=\left(\tfrac{15}{4}-1/(4y^3)+1/(4y^2)-9/(4y)+15y/(4(1-y))\right)\ast\widehat{S}(-2,0,-1,0)$$
$$+\left(-\tfrac{29}{2}+1/(2y^3)-1/(2y^2)+11/(2y)-45y/(2(1-y))\right)\ast\widehat{S}(-2,0,-1,1)$$
$$+\left(\tfrac{15}{4}-1/(4y^3)-3/(4y^2)+7/(4y)+15y/(4(1-y))\right)\ast\widehat{S}(-2,1,-1,0)$$
$$+\left(-15-1/y^3+1/y^2-3/y-15y/(1-y)\right)\ast\widehat{S}(-2,1,-1,1)$$
$$+\left(\tfrac{5}{4}+1/(4y^3)+3/(4y^2)+5/(4y)+5y/(4(1-y))\right)\ast\widehat{S}(-2,2,-1,0)$$
$$+\left(-\tfrac{5}{2}+1/(2y^3)-1/(2y^2)-5/(2y)-5y/(2(1-y))\right)\ast\widehat{S}(-2,2,-1,1)$$
$$+\left(\tfrac{5}{4}+1/(4y^3-9/(4y^2)+5/(4y)+5y/(4(1-y))\right)\ast\widehat{S}(-2,-1,-1,0)$$
$$+\left(-10+2/y^2+6/y-10y/(1-y)\right)\ast\widehat{S}(-2,-1,-1,1)$$
$$+\left(-1/y^4+2/y^3-2/y^2+4/y\right)\ast\widehat{S}(-1,0,-2,1)$$

$$+ (- \tfrac{45}{4} - 3/y^4 + 31/(4y^3) - 15/(4y^2) - 61/(4y) - 45y/(4(1-y)))*\widehat{S}(- 1,0, - 1,0)$$
$$+ (45 - 6/y^4 + 15/y^3 - 15/y^2 + 33/y + 45y/(1-y))*\widehat{S}(- 1,0, - 1,1)$$
$$+ (1/(2y^4) + 1/y^2)*\widehat{S}(- 1,1, - 2,1)$$
$$+ (- 5 + 3/(2y^4) - 3/y^3 - 4/y^2 - 5/y - 5y/(1-y))*\widehat{S}(- 1,1, - 1,0)$$
$$+ (10 + 3/y^4 - 6/y^3 + 6/y^2 + 10/y + 10y/(1-y))*\widehat{S}(- 1,1, - 1,1)$$
$$+ (- 18 + 10/y - 30y/(1-y))*\widehat{S}(- 1, - 2,0,0) + (44 - 20/y + 100y/(1-y))*\widehat{S}(- 1, - 2,0,1)$$
$$+ (- 24 - 120y/(1-y))*\widehat{S}(- 1, - 2,1,0) + (16 + 80y/(1-y))*\widehat{S}(- 1, - 2,1,1)$$
$$+ (- \tfrac{5}{4} - 1/(4y^3) + 9/(4y^2) - 5/(4y) - 5y/(4(1-y)))*\widehat{S}(- 1, - 2, - 1,0)$$
$$+ (10 - 2/y^2 - 6/y + 10y/(1-y))*\widehat{S}(- 1, - 2, - 1,1)$$
$$+ (- 42 + 6/y - 30y/(1-y))*\widehat{S}(- 1, - 1,0,0) + (- 72 - 120y/(1-y))*\widehat{S}(- 1, - 1,1,0)$$
$$+ (1/(2y^4) - 4/y^3 + 5/y^2)*\widehat{S}(- 1, - 1, - 2,1)$$
$$+ (- \tfrac{15}{2} + 3/(2y^4) - 9/(2y^3 + 29/(2y^2) - 35/(2y) - 15y/(2(1-y)))*\widehat{S}(- 1, - 1, - 1,0)$$
$$+ (45 + 3/y^4 - 9/y^3 + 1/y^2 + 5/y + 45y/(1-y))*\widehat{S}(- 1, - 1, - 1,1),$$

$$B_8 = (\tfrac{15}{4} + 3/(2y^3) - 9/(4y^2) - 9/(4y) + 15y/(4(1-y)))*\widehat{S}(- 2,0, - 1,0)$$
$$+ (- \tfrac{37}{2} + 3/(2y^3) - 8/y^2 + 35/(2y) - 45y/(2(1-y)))*\widehat{S}(- 2,0, - 1,1)$$
$$+ (\tfrac{15}{4} - 3/(4y^3) - 3/(4y^2) + 15/(4y) + 15y/(4(1-y)))*\widehat{S}(- 2,1, - 1,0)$$
$$+ (- 15 - 3/(2y^3) + 6/y^2 - 12/y - 15y/(1-y))*\widehat{S}(- 2,1, - 1,1)$$
$$+ (\tfrac{5}{4} + 1/(2y^3) + 5/(4y^2) + 5/(4y) + 5y/(4(1-y)))*\widehat{S}(- 2,2, - 1,0)$$
$$+ (- \tfrac{5}{2} + 1/(2y^3) - 2/y^2 - 5/(2y) - 5y/(2(1-y)))*\widehat{S}(- 2,2, - 1,1)$$
$$+ (\tfrac{5}{4} - 1/(2y^3) + 7/(4y^2) - 11/(4y) + 5y/(4(1-y)))*\widehat{S}(- 2, - 1, - 1,0)$$
$$+ (6 - 1/(2y^3) + 4/y^2 - 7/y - 10y/(1-y))*\widehat{S}(- 2, - 1, - 1,1)$$
$$+ (- \tfrac{45}{4} - 2/y^3 + 21/(4y^2) - 13/(4y) - 45y/(4(1-y)))*\widehat{S}(- 1,0, - 1,0)$$
$$+ (45 - 5/(2y^3) + 2/y^2 + 8/y + 45y/(1-y))*\widehat{S}(- 1,0, - 1,1)$$
$$+ (- 5 + 1/(2y^3) - 3/y^2 - 5/y - 5y/(1-y))*\widehat{S}(- 1,1, - 1,0)$$
$$+ (10 + 3/y^2 + 10/y + 10y/(1-y))*\widehat{S}(- 1,1, - 1,1)$$
$$+ (- 26 - 30y/(1-y))*\widehat{S}(- 1, - 2,0,0) + (44 + 4/y + 100y/(1-y))*\widehat{S}(- 1, - 2,0,1)$$
$$+ (- 48 - 120y/(1-y))*\widehat{S}(- 1, - 2,1,0) + (16 + 80y/(1-y))*\widehat{S}(- 1, - 2,1,1)$$
$$+ (- \tfrac{5}{4} + 5/(4y^2) - 5/(4y) - 5y/(4(1-y))*\widehat{S}(- 1, - 2, - 1,0)$$
$$+ (10 - 1/(2y^3) + 2/y^2 - 5/y + 10y/(1-y))*\widehat{S}(- 1, - 2, - 1,1)$$
$$+ (- 14 + 2/y - 30y/(1-y))*\widehat{S}(- 1, - 1,0,0) + (- 72 - 120y/(1-y))*\widehat{S}(- 1, - 1,1,0)$$
$$+ (- \tfrac{15}{2} + 3/(2y^3) - 11/(2y^2) + 17/(2y) - 15y/(2(1-y)))*\widehat{S}(- 1, - 1, - 1,0)$$
$$+ (13 + 3/y^3 + 11/y^2 + 13/y + 45y/(1-y))*\widehat{S}(- 1, - 1, - 1,1),$$

$$B_9 = (- \tfrac{15}{4} - 3/(4y^4) + 7/(4y^3) + 1/(4y^2) + 1/(4y) - 15y/(4(1-y)))*\widehat{S}(- 2,0, - 1,0)$$
$$+ (\tfrac{45}{2} - 3/(2y^4) + 9/(2y^3) - 15/(2y^2) - 3/(2y) + 45y/(2(1-y)))*\widehat{S}(- 2,0, - 1,1)$$
$$+ (- \tfrac{15}{4} + 3/(4y^4) + 3/(4y^3) - 3/(4y^2) - 15/(4y) - 15y/(4(1-y)))*\widehat{S}(- 2,1, - 1,0)$$
$$+ (15 + 3/(2y^4) - 3/y^3 + 15/y + 15y/(1-y))*\widehat{S}(- 2,1, - 1,1)$$
$$+ (- \tfrac{5}{4} - 1/(4y^4) - 3/(4y^3) - 5/(4y^2) - 5/(4y) - 5y/(4/(1-y)))*\widehat{S}(- 2,2, - 1,0)$$
$$+ (\tfrac{5}{2} - 1/(2y^4) + 1/(2y^3) + 5/(2y^2) + 5/(2y) + 5y/(2(1-y)))*\widehat{S}(- 2,2, - 1,1)$$
$$+ (- \tfrac{5}{4} + 1/(4y^4) - 7/(4y^3) + 15/(4y^2) - 5/(4y) - 5y/(4(1-y)))*\widehat{S}(- 2, - 1, - 1,0)$$
$$+ (10 + 1/(2y^4) - 2/y^3 + 3/y^2 - 10/y + 10y/(1-y))*\widehat{S}(- 2, - 1, - 1,1)$$
$$+ (\tfrac{45}{4} + 5/(4y^4) - 9/(4y^3) - 11/(4y^2) + 45/(4y) + 45y/(4(1-y)))*\widehat{S}(- 1,0, - 1,0)$$
$$+ (- 45 + 5/(2y^4) - 5/y^3 + 10/y^2 - 45/y - 45y/(1-y))*\widehat{S}(- 1,0, - 1,1)$$

TABLE V. (*Continued*.)

$$
\begin{aligned}
& +(5+1/y^3+5/y^2+5/y+5y/(1-y))\ast\widehat{S}(-1,1,-1,0) \\
& +(-10-10/y^2-10/y-10y/(1-y))\ast\widehat{S}(-1,1,-1,1) \\
& +(30-10/y^2+18/y+30y/(1-y))\ast\widehat{S}(-1,-2,0,0) \\
& +(-92+4/y-100y/(1-y))\ast\widehat{S}(-1,-2,0,1)+(88+8/y+120y/(1-y))(-1,-2,1,0) \\
& +(48-80y/(1-y))\ast\widehat{S}(-1,-2,1,1) \\
& +(\tfrac{5}{4}+1/(4y^4)-9/(4y^3)+5/(4y^2)+5/(4y)+5y/(4(1-y)))\ast\widehat{S}(-1,-2,-1,0) \\
& +(-10+1/(2y^4)-2/y^3+11/y^2-10/y-10y/(1-y))\ast\widehat{S}(-1,-2,-1,1) \\
& +(30-10/y^2+10/y+30y/(1-y))\ast\widehat{S}(-1,-1,0,0) \\
& +(120+24/y+120y/(1-y))\ast\widehat{S}(-1,-1,1,0) \\
& +(\tfrac{15}{2}-3/(2y^4)+13/(2y^3)-25/(2y^2+15/(2y)+15y/(2(1-y)))\ast\widehat{S}(-1,-1,-1,0) \\
& +(-45-3/y^4+9/y^3-1/y^2-5/y-45y/(1-y))\ast\widehat{S}(-1,-1,-1,1), \\
B_4 = {} & -B_2+(1/y^3-2/y^2)\ast\widehat{S}(-1,0,-1,0) \\
& +(2/y^3-6/y^2+8/y)\ast\widehat{S}(-1,0,-1,1)+(-1/(2y^3))\ast\widehat{S}(-1,1,-1,0) \\
& +(-1/y^3+2/y^2)\ast\widehat{S}(-1,1,-1,1)+(-12+10/y)\ast\widehat{S}(-1,-1,0,0) \\
& -8\ast\widehat{S}(-1,-1,1,0)+(-1/(2y^3)+4/y^2-4/y)\ast\widehat{S}(-1,-1,-1,0) \\
& +(16-1/y^3+4/y^2-16/y)\ast\widehat{S}(-1,-1,-1,1), \\
B_{10} = {} & -B_6+(-1/y^3+4/y)\ast\widehat{S}(-1,0,-1,0)+(-2/y^3+10/y^2-20/y)\ast\widehat{S}(-1,0,-1,1) \\
& +(1/(2y^3)+1/y^2)\ast\widehat{S}(-1,1,-1,0)+(1/y^3-4/y^2)\ast\widehat{S}(-1,1,-1,1) \\
& +(8-2/y)\ast\widehat{S}(-1,-1,0,0)+24\ast\widehat{S}(-1,-1,1,0) \\
& +(1/(2y^3)-3/y^2+4/y)\ast\widehat{S}(-1,-1,-1,0) \\
& +(-16+1/y^3-6/y^2+12/y)\ast\widehat{S}(-1,-1,-1,1).
\end{aligned}
$$

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[10]A tadpole diagram is a Feynman diagram in which all external lines emanate from one vertex and all internal lines are propagators of massless particles. Conservation of momentum dictates that the total momentum carried by the external lines must be zero. The conventional argument for assigning a zero value to tadpole diagrams is that one could not find a dimensionful scale to express such a diagram were its value finite. It is clear that this argument has at least one loophole: a dimensionless diagram certainly could have a finite (constant) value. By analogy, primal integrals with $K=0$ may be called tadpole integrals. Since the new analytic method reveals that such integrals are but a special subclass of a much larger class of vanishing (in a certain limit) integrals, we call the whole class of integrals tadpolelike integrals or "tadpoles." They are identified by satisfying either or both of the conditions (11).

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[15]On the other hand, having different sets of integrals on the two sides of (4) does not necessarily imply the two sides are unequal. They would be unequal only if it could be proven that among the different integrals those on one side were independent of those on the other. We have not made an attempt to rigorously assure that the integrals used to express B_i are from an independent set, but have simply reduced them to a simple form using straightforward algebraic identities.

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