Hopf Algebra, Complexification of $\mathcal{U}_q(sl(2,\mathbb{C}))$ and Link Invariants*

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When q is explicitly complexified, the q-analogue of the universal enveloping algebra of $sl(2,\mathbb{C})$, $\mathcal{U}_q(sl(2,\mathbb{C}))$, supports a bi-algebra structure besides the one discussed recently by Drinfeld, Jimbo, Reshetikhin and others. The bi-algebra may be a Hopf algebra and has the Hopf algebra of DJR as the limit $q \to$ real. Solutions for the quantum Yang-Baxter equation, representations for Artin's braid group B_n, and link invariants can be derived from representations of the bi-algebra in $\mathcal{U}^*_q(sl(2,\mathbb{C}))$. The classical Alexander-Conway link polynomial is given by the two-dimensional representation. The complex link invariants of Lee, Couture & Schmeing are associated with higher dimensional representations of $\mathcal{U}^*_q(sl(2,\mathbb{C}))$. A proposition giving the link invariant as an abstract map $\mathcal{U}_q^{\otimes n} \to e$, where e is the identity in $\mathcal{U}_q$, is stated. It contrasts with, and has a more general application than, the usual definition via the Markov trace.

1. Introduction

Recently interest in the solutions of the Yang-Baxter equation[1] has increased considerably. The Yang-Baxter equation played a key role in the quantum inverse scattering method and solvable state models. Subsequently it was realized that, because the solutions give representations to

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Artin's braid group B_n, they can be used to construct link invariants. As well, they are closely related to monodromies of the correlation functions of conformal field theories in two dimensions.

Drinfeld[2] and Jimbo[3] showed that underlying the solutions to the Yang-Baxter equations is a vast mathematical structure: to each simple or Kac-Moody algebra $\mathfrak{g}$, the q-deformation of the universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ has a Hopf algebra structure, and to each representation of the Hopf algebra there is a solution to the Yang-Baxter equation. The representation theory of this structure was recently studied in detail by Reshetikhin[4] who derived, among other things, the fusion rules for the R-matrices associated with solutions of the Yang-Baxter equation, and showed how link invariants are constructed.

In the algebraic structure discussed above, the link invariant associated with the lowest, or two-dimensional representation of $\mathcal{U}_q(sl(2,\mathbb{C}))$ is just the Jones[5] polynomial, that associated with the three and higher dimensional representations of $\mathcal{U}_q(sl(2,\mathbb{C}))$ are the link invariants of Akutsu, Wadati and co-workers[6]. These higher order link invariants and others similarly derived (see also those derived by Yamagishi, Ge and Wu[7] via the three-dimensional topologically invariant gauge theory of Witten[8]) are generalizations of Jones' polynomial. They share a common characteristic that sets them apart from the classical Alexander-Conway polynomial[9]: the Alexander-Conway maps all split links to the null-polynomial, whereas the Jones polynomial and its generalizations do not. On the other hand, it is well-known that the Jones polynomial and the Alexander-Conway polynomial are closely related. They are respectively one-variable specializations[10] of the two-variable link invariant of HOMFLY[11]. To illustrate another relation, denote $\{L\}$ as the set of all links, $\mathbb{P}$ the space of power series in t, $V_J(t)$ the Jones polynomial, $P_{AC}(t,\omega=-1)$ the Alexander-Conway polynomial. Then $V_J(t)$ and $P_{AC}(t,\omega=-1)$ are distinct maps of $\{L\} \to \mathbb{P}$; $V_J(-1) = P_{AC}(-1,\omega=-1)$ is a map $\{L\} \to \mathbb{C}$ which maps all split links to zero; $V_J(1) = P_{AC}(1;\omega=1)$ is a map $\{L\} \to \mathbb{C}$ that does not map all split links to zero. The extra parameter ω is characteristic of the Alexander-Conway class of link invariants. Higher members of this class were recently found by Lee & Couture[12] and Lee, Couture & Schmeing[13]. In particular, the 3-state member of the class $P_{LC}(t;\omega =$

$e^{2\pi i/3}$) is related to the 3-state Akutsu-Wadati polynomial $V_{AW}(t)$ the same way P_{AC} is related to V_J.

The P_{LC} invariants were found by directly solving the Yang-Baxter equation. Since they are so closely related to the link invariants of $\mathcal{U}_q(sl(2,\mathbb{C}))$, an intriguing question is whether the Alexander-Conway class of invariants has anything to do with $\mathcal{U}_q(sl(2,\mathbb{C}))$, and if so, in what way. The answer is now known: the class is directly associated with a bi-algebra structure in the complexified $\mathcal{U}_q(sl(2,\mathbb{C}))$ which specializes to the algebra of Drinfeld & Jimbo in the real limit.

For the rest of this paper, in Sec. 2 we introduce the concept of Hopf algebra by explicitly constructing representations for it in terms of 2×2 matrices. We find that only two distinct representations exists if each one is to give a solution to the Yang-Baxter equation. In sec. 3 we briefly review some of the results of Drinfeld and Jimbo, following mostly Reshetikhin. We identify one of the representations derived in the previous section as precisely the lowest dimensional representation for Drinfeld & Jimbo's Hopf algebra for $\mathcal{U}_q(sl(2,\mathbb{C}))$. The other represents a generalization. The new algebra, which we shall call a pseudo-Hopf algebra, its antipode is not yet specified, also lies in the exponentiation of $\mathcal{U}_q(sl(2,\mathbb{C}))$, but with q explicitly and necessarily complex. We then give the 3-state representations of both algebras. The representation for the Drinfeld-Jimbo algebra is of course known. The representation for the pseudo-Hopf algebra gives precisely the complex solution for the Yang-Baxter equation from which the link invariant $P_{LC}(t;\omega)$ mentioned earlier is constructed. In sec. 4 we discuss the construction of link invariants and state a proposition with which the link invariant can be defined via an abstract map

$$\mathcal{U}_q(\mathfrak{g})^{\otimes n} \to e \in \mathcal{U}_q(\mathfrak{g}) \tag{1.1}$$

The proposition is derived empirically, but not proven. The map differs from the well-known map

$$\mathrm{End}(V^{\otimes n}) \to \mathbb{C} \tag{1.2}$$

or its abstract form

$$\mathcal{U}_q(\mathcal{g})^{\otimes n} \rightarrow \mathbb{C} \tag{1.3}$$

used in the Markov-trace construction of link invariants, where V is the vector space carrying the representation of $\mathcal{U}_q(g)$. Of course (1.3) follows from (1.1), but not vice versa. We believe (1.1) is the true algebraic definition of link invariants. It is known that (1.3) is useless when q is an integral root of unity, because the kernel of the map is its entire domain. This is also true for all known representations of the new algebra. Conversely, (1.1) works for both cases. I thank Peter Leivo and Michel Couture, with whom I collaborated on most of the work reported here.

2. Hopf Algebra: Two Toy Representations

Let A be an algebra with elements a and basis e_σ. By a basis one means that each element $a \in A$ can be expressed as a linear combination of the e_σ's. A is a Hopf algebra if:

(i) There are homomorphic maps such that

multiplication m: $A \otimes A \rightarrow A$ defined by the coefficients $m^\tau_{\rho\sigma}$

$$m(e_\rho, e_\sigma) \equiv e_\rho e_\sigma = m^\tau_{\rho\sigma} e_\tau \tag{2.1}$$

co-multiplication Δ: $A \rightarrow A \otimes A$ defined by the coefficients $\mu^{\rho\sigma}_\tau$

$$\Delta(e_\tau) = \mu^{\rho\sigma}_\tau \, e_\rho \otimes e_\sigma \tag{2.2}$$

co-unit ε: $A \rightarrow \mathbb{C}$ defined by the coefficients c_σ

$$\varepsilon(e_\sigma) = c_\sigma \tag{2.3}$$

(ii) There is an antiautomorphic map

antipode γ: $A \rightarrow A$ defined by the coefficients γ^σ_ρ

$$\gamma(e_\rho) = \gamma^\sigma_\rho \, e_\sigma \tag{2.4}$$

For each $a \in A$ the associativity relations hold

(iii) $$(\Delta \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \Delta)\Delta(a) \tag{2.5}$$

(iv) $$(\varepsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varepsilon)\Delta(a) = a \tag{2.6}$$

(v) $$m(\mathrm{id} \otimes \gamma)\Delta(a) = m(\gamma \otimes \mathrm{id})\Delta(a) = \varepsilon(a)e \tag{2.7}$$

Recall that a homomorphism preserves the order of multiplication, for example

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \forall a,b \in A.$$

On the other hand, an antimorphism reverses the order of multiplication,

$$\gamma(a,b) = \gamma(b)\gamma(a), \qquad \forall a,b \in A.$$

To illustrate what the constraints (2.5) and (2.7) are, let $a=e_\sigma$, then the left-hand side of (2.5) becomes, in component form,

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(e_\sigma) &= \mu_\sigma^{\rho\tau}(\Delta \otimes \mathrm{id})e_\rho \otimes e_\tau = \mu_\sigma^{\rho\tau}\Delta(e_\rho) \otimes e_\tau \\ &= \mu_\sigma^{\rho\tau}\mu_\rho^{\mu\upsilon}e_\mu \otimes e_\upsilon \otimes e_\tau\end{aligned}$$

similarly for the right-hand side. Thus in full detail (2.5) and (2.7) give the constraints

$$\mu_\sigma^{\rho\tau}\ \mu_\rho^{\mu\upsilon} = \mu_\sigma^{\mu\rho}\ \mu_\rho^{\upsilon\tau} \tag{2.8}$$

$$\mu_\sigma^{\rho\tau}\ \gamma_\tau^\upsilon\ m_{\rho\upsilon}^\mu\ e_\mu = \mu_\sigma^{\rho\tau}\ \gamma_\rho^\upsilon\ m_{\upsilon\tau}^\mu\ e_\mu = c_\sigma e \tag{2.9}$$

The last equation suggests the notation $(\mu_\sigma\gamma)^{\rho\upsilon} \equiv \mu_\sigma^{\rho\tau}\gamma_\tau^\upsilon$, $(\gamma m^\mu)_{\rho\tau} \equiv \gamma_\rho^\upsilon\ m_{\upsilon\tau}^\mu$, so that (2.9) can be compact expressed as

$$\mathrm{Tr}(\mu_\sigma\gamma(m^\rho)^\top)e_\rho = \mathrm{Tr}((\mu_\sigma)^\top\gamma m^\rho)e_\rho = c_\sigma e \tag{2.10}$$

suggesting that μ_σ and m^σ have a duality relation. This suggestion is reinforced by the similarity between (2.8) and the following relation derived from the associativity of multiplication of the product $e_\mu e_\upsilon e_\tau$

$$m_{\mu\rho}^\sigma\ m_{\upsilon\tau}^\rho = m_{\mu\upsilon}^\rho\ m_{\rho\tau}^\sigma \tag{2.11}$$

We thus have the following:

A Hopf algebra $\tilde{A}$ with basis e^{σ} is dual to the Hopf algebra A if its multiplication $\tilde{m}$ is defined by $\mu_{\tau}^{\rho\sigma}$

$$\tilde{m}(e^{\rho},e^{\sigma}) = \mu_{\tau}^{\rho\sigma}\, e^{\tau} \tag{2.12}$$

and its co-multiplication $\tilde{\Delta}$ is defined by the transpose of $m_{\rho\sigma}^{\tau}$

$$\tilde{\Delta}(e^{\tau}) = m_{\rho\sigma}^{\tau}\, e^{\sigma} \otimes e^{\rho} \tag{2.13}$$

The antipode $\tilde{\gamma}$ and co-unit $\tilde{\varepsilon}$ of $\tilde{A}$ is then determined by (2.7). Let $\mathcal{U}$ be the union of A and $\tilde{A}$, $\mathcal{U} \equiv A \cup \tilde{A}$. Then for elements in the intersection of A and $\tilde{A}$, the two maps in each of the pairs, m and $\tilde{m}$, Δ and $\tilde{\Delta}$, γ and $\tilde{\gamma}$, ε and $\tilde{\varepsilon}$ must be identical, respectively. Otherwise the maps need not be the same. Therefore no confusion will arise if only one set of symbols m, Δ, γ, ε are used for the abstract maps. Thus

$$\Delta(e^{\sigma}) = m_{\tau\rho}^{\sigma}\, e^{\rho} \otimes e^{\tau},\ \gamma(e_{\sigma}) = \gamma_{\sigma}^{\rho} e_{\rho},\ \gamma(e^{\rho}) = \tilde{\gamma}_{\sigma}^{\rho}\, e^{\sigma}, \text{ etc.}$$

Although a Hopf algebra and its dual appear to be tightly structured, a *representation* for it is not so difficult to find. We shall now construct one explicitly in terms of 2×2 matrices which obey the usual matrix multiplication rules. There are only four linearly independent 2×2 matrix elements

$\lambda_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $\lambda_2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $\lambda_- \equiv \lambda_4 \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $\lambda_+ \equiv \lambda_3 \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

which separate naturally into three sets, the commuting, or Cartan sector including λ_1 and λ_2; the noncommuniting raising operator λ_+; the noncommuting lowering operator λ_-. The multiplication rules are

$$\lambda_i\lambda_j = \delta_{ij}\lambda_i \qquad i,j = 1,2 \tag{2.14}$$

$$\lambda_2\lambda_- = \lambda_-\lambda_1 = \lambda_-; \qquad \lambda_1\lambda_+ = \lambda_+\lambda_2 = \lambda_+ \tag{2.15,16}$$

$$\lambda_-\lambda_+ = \lambda_2; \qquad \lambda_+\lambda_- = \lambda_1 \tag{2.17}$$

With all other multiplications giving zero. Note the sets $\{\lambda_1,\lambda_2,\lambda_+\}$ and $\{\lambda_1,\lambda_2,\lambda_-\}$ form subalgebras. Furthermore, assigning the charge ±1 to $\lambda_{\pm}$, and 0 to λ_1 and λ_2, then the multiplications (2.14-17) are charge conserving. We now choose $\{\lambda_1, \lambda_2, \lambda_+\}$ as the basis of the Hopf algebra $\tilde{A}$

$$e^1 = \lambda_1,\ e^2 = \lambda_2,\ e^3 = \lambda_+ \tag{2.18}$$

The most general charge conserving basis for its dual A is

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.19}$$

or simply $e_i = (\alpha\lambda)_i$, $i = 1,2$, where α is a 2×2 matrix restricted to the Cartan sector, and

$$e_3 = \eta\lambda_- \tag{2.20}$$

The c-numbers α_{ij} and η are to be determined by the fact that $\tilde{A}$ is dual to A. Note that e^1 and e^2 respectively are their own inverses, e_3 and e_4 do not have inverses, while

$$e_i^{-1} = (\bar{\alpha}\lambda)_i; \quad \bar{\alpha}_{ij} \equiv (\alpha_{ij})^{-1} \tag{2.21}$$

Recall that the co-multiplication for A is determined by the multiplication for $\tilde{A}$. Therefore, from (2.14) and (2.16), the (*representations* of) $\Delta(e_\sigma)$ are

$$\Delta(e_i) = e_i \otimes e_i \qquad\qquad i = 1,2 \tag{2.22}$$

$$\Delta(e_3) = e_3 \otimes e_2 + e_1 \otimes e_3 \tag{2.23}$$

Multiplications of e_σ are given by (2.19,20) and (2.14,16):

$$e_i e_j = \alpha_{il}\alpha_{jl}(\alpha^{-1})^{lk}\, e_k; \qquad\qquad e_3^2 = 0 \tag{2.24}$$

$$e_i e_3 = \alpha_{i2} e_3; \qquad\qquad e_3 e_i = \alpha_{ji} e_3 \tag{2.25}$$

These in turn determine the co-multiplication of e^σ, or equivalently, λ_1, λ_2 and λ_+:

$$\Delta(e^k) = \Delta(\lambda_k) = \alpha_{il}\alpha_{jl}(\alpha^{-1})^{lk}\, e^i \otimes e^j, \qquad k = 1,2 \tag{2.26}$$

$$\Delta(e^3) = \Delta(\lambda_+) = \alpha_{i2}\, e^3 \otimes e^i + \alpha_{1i}\, e^i \otimes e^3 \tag{2.27}$$

Since e_i is a linear combination of λ_i's, $\Delta(\lambda_\mu)$ is also determined by (2.22):

$$\Delta(\lambda_h) = (\alpha^{-})_{kl}\alpha_{li}\alpha_{lj}\, e^i \otimes e^j \tag{2.28}$$

Therefore α *must be symmetric*. In this case (2.23) and (2.27) become

$$\Delta(\lambda_\pm) = \lambda_\pm \otimes e_2 + e_1 \otimes \lambda_\pm \tag{2.29}$$

Note that (2.8) is guaranteed by the associativity of multiplication for $\breve{A}$. There remains only the task to construct the antipode and the co-unit. The solutions of (2.10) and its dual

$$\mathrm{Tr}(\mu_\rho \tilde{\gamma} m^\sigma)e^\rho = \mathrm{Tr}(\mu_\rho \tilde{\gamma} m^\sigma)e^\rho = c^\sigma e \tag{2.30}$$

give γ, $\tilde{\gamma}$ as 3×3 block-diagonal matrices:

$$\gamma = \left(\begin{array}{c|c} \bar{\alpha}\alpha^{-1} & \\ \hline & -\alpha_{12}^{-2} \end{array}\right); \qquad \tilde{\gamma} = \left(\begin{array}{c|c} \alpha^{-1}\bar{\alpha} & \\ \hline & -\alpha_{11}^{-1}\,\alpha_{22}^{-1} \end{array}\right) \tag{2.31}$$

and the co-units

$$\varepsilon \begin{pmatrix} e_i \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \varepsilon \begin{pmatrix} e^i \\ e^3 \end{pmatrix} = \begin{pmatrix} \sum_j (\alpha^{-1})^{ji} \\ 0 \end{pmatrix} \tag{2.32}$$

One can check that (2.5) and (2.6) are also satisfied. This completes the 2×2 representation of an algebra which appears to have a Hopf algebra structure. Note that the representations of all the nontrivial maps are given in terms of matrix elements of α, which is itself surprisingly unrestricted; it is only required that it be symmetric, has no zero elements (so that $\bar{\alpha}$ exists), and has an inverse. We now impose an additional constraint on the algebra structure. Define an element $\mathscr{R} \in A \otimes \breve{A}$ by

$$\mathscr{R} = e_\tau \otimes e^\tau \tag{2.33}$$

and demand that

$$(\sigma \bullet \Delta)(a)\mathscr{R} = \mathscr{R}\Delta(a), \qquad \forall a \in A \cup \breve{A} \tag{2.34}$$

split links only when $t = \omega_3$, where $P_{AW}(\omega_3) = P_{LC}(\omega_3;\omega_3)$.

Finally, we present, minus proof, the following proposition for constructing a link invariant *on the (pseudo) Hopf algebra* $\mathcal{U}$ without reference to any representation. Let $\beta \in B_n$ be a braid with positive (negative) crossings represented by generators g_i (g_i^{-1}). Express β with n downward pointing strings as an element in $\mathcal{U}^{\otimes n}$, each string occupying one $\mathcal{U}$. At each positive (negative) crossing the string descending from the left has a factor $e_\sigma (e^\sigma)$, and the string descending from the right has a factor of $e^\sigma(\gamma(e_\sigma))$. Now close n-1 strings, leaving the right-most open, by connecting, respectively, the n-1 ordered ends at the bottom of the braid to the n-1 ordered ends at the top of the braid. On the algebra, the action of connecting two strings is to insert a factor $h \in \mathcal{U}$ at the connection and then multiply the two elements. The element h is given by $h \equiv e_\sigma \gamma(e^\sigma)$. When one end of a string is connected to its opposite end by closing, thus forming a link, again a h is inserted at the connection, and the connected element corresponding to the link is taken value on $\mathbb{C}$ by the map $\tilde{\varepsilon}: \mathcal{U} \to \mathbb{C}$, defined by $\mathrm{Rep}(\tilde{\varepsilon}(a)) = \mathrm{Trace}(\mathrm{Rep}(a))$, $\forall a \in \mathcal{U}$.

Proposition. The procedure described above for closing n-1 strings of a braid $\beta \in B_n$ is a map

$$\Gamma(\beta) : \mathcal{U}^{\otimes n} \to e \in \mathcal{U}. \tag{4.6}$$

The function $\hat{P}[\beta] \equiv \lambda^{(\Sigma + 1 - n)/2}\, \Gamma(\beta)$ where $\lambda \in \mathbb{C}$ is defined by $e^\sigma h \gamma(e_\sigma) = \lambda e$, is a link invariant.

This is just the abstract form of (4.4) and (4.5).

Theorem 1. $e_\rho h e^\rho = e$. (4.7)

Proof: $e_\rho h e^\rho = e_\rho e_\sigma \gamma(e^\sigma) e^\rho = m^\alpha_{\rho\sigma}\, \tilde{\gamma}^\sigma_\tau\, \mu^{\tau\rho}_\beta\, e_\alpha\, e^\beta$

$$= (m(\gamma \otimes \mathrm{id})\, \Delta(e^\alpha))e_\alpha = \varepsilon(e^\alpha)e_\alpha = e$$

where ε is the co-unit. Note the left-hand side of (4.7) is just the closing of one of the two strings in a braid with one positive crossing. We therefore have:

Lemma 1. For an R-matrix given (4.1), the element h needed to construct the Markov trace or the map (4.6) always exists and is given by $h = e_\sigma \gamma(e^\sigma)$.

Theorem 2. $e^\sigma h\gamma(e_\sigma) = \lambda e$. (4.8)

The left-hand side represents the closing of a braid with one negative crossing; λ is the constant needed in (4.3.5,6) to ensure that the expressions given are invariant under the first Reidemeister or the second Markov move. The proofs of the proposition and theorem 2 will be given elsewhere.

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Note Added: After these lectures were given I received the preprint "Knots, abstract tensors and the Yang-Baxter equation" by L. Kauffman where a description of the Alexander polynomial in terms of a bialgebra is given. I thank Kauffman for sending me the preprint.