ANNALS OF PHYSICS 157, 408–431 (1984)

On the Regularization of a Class of Divergent Feynman Integrals in Covariant and Axial Gauges

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Received August 12, 1983; revised December 14, 1983

A hybrid of dimensional and analytic regularization is used to regulate and uncover a Meijer's G-function representation for a class of massless, divergent Feynman integrals in an axial gauge. Integrals in the covariant gauge belong to a subclass and those in the light-cone gauge are reached by analytic continuation. The method decouples the physical ultraviolet and infrared singularities from the spurious axial gauge singularity but regulates all three simultaneously. For the axial gauge singularity, the new analytic method is more powerful and elegant than the old principal value prescription, but the two methods yield identical infinite as well as regular parts.

1. INTRODUCTION

Dimensional regularization [1–4] is a powerful tool for regulating the ultraviolet [1] and infrared [2] divergent integrals intrinsic to quantum field theories. Because the method preserves gauge invariance and at the same time provides the easiest way to isolate the infinite part as well as the leading logarithmic term of divergent Feynman integrals, from its conception it has been extensively used in the study of renormalization [1–5] and the dominant asymptotic behaviour of gauge theories in perturbation calculations [6].

The analytical properties of a dimensionally regulated integral do not appear to have been fully explored, however, particularly for integrals in an axial gauge [7], which are especially difficult to evaluate. The chief advantage in choosing an axial gauge is that the Faddeev–Popov ghosts [8] that are otherwise required to uphold Ward identities [9] in non-Abelian theories are decoupled from the physical fields. This greatly simplifies calculations and makes practicable otherwise intractable calculations in theories such as quantum gravity. Other advantages of the axial gauge are that it yields mass factorization [10], and in hard quantum chromodynamic processes, a judicious choice of the special planar gauge [11] causes virtual gluons to be effectively physical, i.e., transversely polarized.

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0003-4916/84 $7.50

The price one pays for the convenience of the axial gauge is that integrals involving the propagator now have, in addition to the ultraviolet and infrared divergencies that they may otherwise possess, unphysical or "spurious" singularities that are associated only with the axial gauge. In the literature, integrals that suffer from such axial gauge singularities have been generally (but not exclusively) regularized with the principal value prescription [12]. Recently it has been shown [13, 39] that this prescription, used in conjunction with the method of dimensional regularization, yields well-defined and consistent results for the infinite parts of axial gauge integrals of two-point functions in Yang–Mills theories and quantum gravity at the one-loop level. The prescription, however, is sufficiently cumbersome that the evaluation of any axial gauge integral is a substantial undertaking. Moreover, the evaluation of the finite (or regular) parts of these integrals, other than the leading logarithmic term, is difficult with this prescription.

In this paper we propose a method, based on dimensional regularization (the dimension of space-time is generalized to a continuous variable) *and* analytic regularization (exponents are generalized to continuous variables) [14], for calculating a very general class of massless divergent integrals in the axial gauge; integrals in the covariant gauge, which are free from spurious singularities, constitute a subset of the class. Specifically, in our method spurious singularities are dealt with by analytic regularization, *not* by the principal value prescription. It will be demonstrated that the proposed new method is more powerful and elegant than the old one. At the same time, by means of constructing an axial gauge "regulator" for the principal value prescription it will be shown that both methods yield identical results, for the finite as well as the infinite parts.

On the broader perspective of regularization in general, not restricted to that of axial gauge singularities, we observe that analytic regularization and dimensional regularization, for singularities that can be regularized by the two methods separately, yield identical results, apart from certain terms that can be identified and subtracted. One type of singularity that cannot be regularized by dimensional regularization alone but can be regularized by the other method is the axial gauge singularity. We have not encountered any type of singularity that can be regulated by dimensional regularization but not by analytic regularization. In this sense, at least for the evaluation of the class of integrals considered, dimensional regularization is in fact redundant. It must be emphasized that we *do not* advocate the replacement of dimensional regularization by analytic regularization. The reason is obvious, for in situations where the former method works, it is much the superior one requiring the generalization of only one integer—the number of dimensions—into a continuous variable. The latter method requires the generalization of several integer exponents. On the other hand, because dimensional regularization has some known limitations [1, 3, 4, 15] arising from the ambiguity of doing algebra in continuous dimensions—the most famous one being that related to the Bell–Jackiw–Adler anomaly [16]—the recognition that analytic and dimensional regularization are equivalent, as far as regulating integrals is concerned, is important; since analytic regularization does not affect the algebra, it is clear that (for situations where dimensional regularization will work) one should do all the algebra in four dimensions to reduce the integrand to a function of scalars in Euclidean (or Minkowski) space before regulating the integral by dimensional regularization. This is precisely the strategy adopted in the recently proposed method of dimensional reduction [17]. This being the case, we further demonstrate that it is unnecessary to restrict this method to spaces of less than four dimensions [17]. The fact that we regulate only Feynman integrals also allows us to demonstrate that our analytic regularization preserves gauge invariance. This is in contrast to Speer's [14] analytic method of regulating propagators which has the appearance of not preserving gauge invariance [3]. In this paper we shall concentrate on understanding the formal properties of analytic regularization. The subject of gauge invariance will be dealt with separately [18], when it will be demonstrated explicitly that analytically regularized radiative corrections satisfy Ward identities.

The class of integrals we shall study is defined by

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) \equiv \int d^{2\omega}q[(p-q)^2]^{\kappa}(q^2)^{\mu}(q \cdot n)^{2\nu+s}, \tag{1.1}$$

where ω, κ, μ, ν are arbitrary, *continuous* variables, $s = 0$ or 1, p is an external momentum and n is an external vector used to define the axial gauge condition $A \cdot n = 0$; A is the gauge field. For simplicity we choose to work in a (2ω-dimensional) Euclidean space; Minkowski space is reached by analytic continuation [19]. Whenever the situation allows, we shall suppress the subscript and/or some of the variables of the function on the left-hand side of (1.1). Thus we may write $S_{2\omega}(p, n)$, $S(p, n)$, or simply S, which we shall call an S-integral. The class of integrals (1.1) is the generalization of the class of "primal" four-dimensional integrals $S_4(p, n; K, M, N, s)$ with integer exponents. Our main result is the discovery of a closed-form expression for the S-integrals that is a well-defined and analytic function of ω, κ, μ, ν and the scalar products p^2, $p \cdot n$ and n^2.

When $\omega = 2$ and κ, μ and ν are integers, the S-integrals reduce to primal integrals in perturbation calculations at the one-loop level for two-point functions in massless Yang–Mills theories and quantum gravity [13]. The subset with $\nu = s = 0$ are the corresponding integrals in covariant gauges. Our motivation for letting the exponents κ, μ and ν be continuous is:

(a) It is necessitated by the method of analytic regularization.

(b) Having κ, μ and ν continuous allows us to generate and regulate integrals with integrands containing powers of $\ln(p-q)^2$, $\ln q^2$ and $\ln(q \cdot n)^2$, by taking partial derivatives of the S-integral with respect to κ, μ and ν, respectively. Such integrals arise in multi-loop calculations.

(c) Integrals with noninteger exponents may appear in nonperturbation calculations even when they do not appear in perturbation calculations [20].

If our sole purpose were to regulate the axial gauge singularity (by analytic regularization) it would only be necessary to generalize the exponent ν; singularities associated with the exponents κ and μ can be more expediently regulated by dimensional regularization. However, by generalizing all κ, μ and ν we are able to establish the relation between analytic regularization, dimensional regularization and dimensional reduction discussed earlier.

The rest of the paper is organized as follows. In Section 2 we present our main result, relating the S-integrals to a Meijer's G-function [21] which is a transcendent of hypergeometric functions and is a well-defined, analytical function of ω, κ, μ, ν and $y \equiv (p \cdot n)^2/p^2n^2$. The derivation, details of which are given in two Appendices, is naturally divided into two steps: the first regulates the S-integral to "canonical" form (Appendix A), and the second identifies the canonical integral as a Meijer's G-function (Appendix B). The divergent nature of the primal S-integral is revealed in the contour integral representation of the G-function by pinches of the contour at certain values of the variables. It will be shown that the infinite part is a certain power of p^2 times a terminating polynomial in y, and the finite or regular part is the sum of a terminating polynomial plus an explicit series in y if $|y| \leqslant 1$, or a different series in $1/y$ if $|y| > 1$, plus logarithmic terms. In the case of covariant gauges, i.e., when $\nu = s = 0$, all series collapse to a form independent of y, as expected, since this integral must be independent of n. The G-function representation treats the cases of space-like ($n^2 < 0$) and time-like ($n^2 > 0$) gauges equally well. Two special gauges, corresponding to the limits $y = 0$ and $y \to +\infty$, are treated as special cases of the continuation.

In Section 3 and Appendix C we show that our analytic regularization of axial gauge singularities yields a result which is identical to that given by the principal value prescription [12]. We show, by explicit construction, that the principal value prescription for an integral with axial gauge singularities of arbitrary order yields a result that can be compactly expressed as a polynomial in a differential operator operating on a sum of G-functions.

To illustrate the power of the G-function representation, in Section 4 we present several analytic examples. The G-function representation also provides new insights concerning covariant gauge integrals and tadpole integrals. We classify all primal S-integrals and present their infinite and finite parts compactly in Table I. Section 5 is a summary.

2. Analytic Representation of the S-Integral

2.1. *General Considerations*

The integral under scrutiny may be formally treated as a function of several complex variables. To justify this approach, consider the integral defined by

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \int d^{2\omega}q(q^2)^{\mu}(q \cdot n)^s[(q \cdot n)^2]^{\nu}[(p-q)^2]^{\kappa}, \tag{2.1}$$

where the integration extends over a Euclidean space generalized to 2ω dimensions [1, 3, 4] in a manner discussed in Appendix A. To guarantee that this integral has meaning, it suffices to choose (continuous) ω compatible with arbitrary (continuous) variables (μ, ν and κ with $s=0$ or 1) such that the integral representation (2.1) exists. In comparison and contrast to 'tHooft and Veltman [1], who regularize only ultraviolet divergencies, it is not sufficient to demand that $\mathrm{Re}(\omega)$ be arbitrarily large and negative; in the definition (2.1) there exist infrared and axial gauge (spurious) singularities with which to contend. However, a region in (ω, μ, ν, κ)-space exists such that (2.1) is well defined. So, it is enough [22] to devise a representation for the integral (2.1) valid for a larger range of the variables but with some overlap with the region of existence.

In Appendices A and B, the following result is derived:

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa+\nu}(n^2)^{\nu}\,\Gamma(s+\nu+\frac{1}{2})(p\cdot n)^s}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)\,\Gamma(2\omega+2\nu+\mu+\kappa+s)} \times G^{2,3}_{3,3}\left(y \,\middle|\, \begin{matrix} 1-\omega-\mu-\nu-s, \omega+\mu+\nu+\kappa+1, \nu+1; \\ 0, \omega+\nu+\kappa; \frac{1}{2}-s \end{matrix}\right), \tag{2.2}$$

where $y = (p\cdot n)^2/p^2n^2$, and G is Meijer's G-function [21], a compact notation for a function which can be represented either as a contour integral (2.7) or as a sum of two generalized hypergeometric functions. In the derivation of (2.2) a number of conditions are required ((B.6) and (A.8)), which collectively delineate the region in which the integral (2.1) exists. The conditions are

$$-\tfrac{1}{2} - s < \mathrm{Re}(\nu) < 0, \tag{2.3a}$$

$$\mathrm{Re}(\mu) < 0, \tag{2.3b}$$

$$\mathrm{Re}(\kappa) < 0, \tag{2.3c}$$

$$|y| \leqslant 1, \tag{2.3d}$$

$$-\mathrm{Re}(\nu) + \mathrm{Max}\left(\mathrm{Re}\left(-s-\mu, -\frac{\mu+\kappa+s}{2}\right)\right) < \mathrm{Re}(\omega) < -\mathrm{Re}(\mu+\nu+\kappa). \tag{2.3e}$$

Of course, the right-hand side of (2.2) is well defined for all values of the variables and the conditions (2.3) may be dispensed with.

Since κ, μ, ν and ω are thought of as being independent (real) variables, it is convenient to introduce some simplifying notation:

$$\kappa = K + \rho, \tag{2.4a}$$

$$\mu = M + \sigma, \tag{2.4b}$$

$$\nu = N + \tau, \tag{2.4c}$$

$$\omega = 2 + \varepsilon, \tag{2.4d}$$

where K, M and N are integers and ρ, σ, τ and ε are variables which will eventually be made arbitrarily small. Furthermore, we define the indices

$$\alpha_0 \equiv -\mu - \nu - s - \omega, \tag{2.5a}$$

$$\alpha_1 \equiv \kappa + \mu + \nu + \omega, \tag{2.5b}$$

$$\alpha_2 \equiv \nu, \tag{2.5c}$$

$$\beta_1 \equiv \alpha_b = \kappa + \nu + \omega, \tag{2.5d}$$

composed of integer parts:

$$A_0 = -M - N - s - 2, \tag{2.5e}$$

$$A_1 = K + M + N + 2, \tag{2.5f}$$

$$A_2 = N, \tag{2.5g}$$

$$B_1 = K + N + 2, \tag{2.5h}$$

and epsilons:

$$\varepsilon_0 = -\alpha_0 + A_0 = \sigma + \tau + \varepsilon, \tag{2.5i}$$

$$\varepsilon_1 = -\alpha_1 + A_1 = -\rho - \sigma - \tau - \varepsilon, \tag{2.5j}$$

$$\varepsilon_2 = -\alpha_2 + A_2 = -\tau, \tag{2.5k}$$

$$\varepsilon_b = \beta_1 - B_1 = \rho + \tau + \varepsilon, \tag{2.5l}$$

in terms of which

$$S = \frac{\pi^{\omega}(p^2)^{\alpha_1}(n^2)^{\alpha_2}(p \cdot n)^s\, \Gamma(\alpha_2 + s + \frac{1}{2})}{\Gamma(\beta_1 - \alpha_0)\,\Gamma(\beta_1 - \alpha_1)\,\Gamma(-\alpha_0 - \alpha_1 - s)\,\Gamma(-\alpha_2)} \times G^{2,3}_{3,3}\left(y \,\middle|\, \begin{matrix} 1+\alpha_0, 1+\alpha_1, 1+\alpha_2; \\ 0, \beta_1; \frac{1}{2} - s \end{matrix}\right). \tag{2.6}$$

The G-function is symmetric under any permutation among α_0, α_1 and α_2. S has less symmetry because of the factors in (2.6) exterior to the G-function; aside from the factor $(p^2)^{\alpha_1}$, S is symmetric under the interchange $\alpha_0 \leftrightarrow \alpha_1$. The factor $(p^2)^{\alpha_1}$ reflects the overall dimension of S save the unimportant factor $(p \cdot n)^s$. From (2.1) and (2.5a–c), the indices α_0, α_1 and α_2 can be recognized as labelling the infrared, ultraviolet and axial gauge singularities, respectively, of the original S-integral, and shall be referred to as such. Significantly, with one exception, ω appears in (2.6) only via the indices of (2.5), i.e., in linear combinations with κ, μ and ν, and always with a relative coefficient ± 1. The exception is the factor π^{ω}, which has no bearing on the singular properties of S; unless otherwise mentioned, we shall ignore this factor in our discussion.

2.2. *Regularization*

Consider the primal integral $S_4(p, n; K, M, N, s)$, whose integral representation (2.1) may or may not exist. The S integral (2.1) with arbitrary parameters is a generalization of the primal integral, and may be analytically continued to all values of the parameters using (2.2). We define the regularized primal integral to be the right-hand side of (2.2) in the limit ε, ρ, σ, $\tau \to 0$.

The regularization process is intimately connected with the manner in which ρ, σ, τ and ε are set to zero. In the first place we wish to regularize the axial gauge singularity $(q \cdot n = 0)$ which lies on the path of integration. To achieve this, we use analytic regularization (A.1) by letting ν become a continuous (complex) variable, and requiring that ν lie in the range (2.3a) in which (2.1) is defined—the axial gauge singularity becomes integrable. In the final result (2.2) we consider values of ν outside the original range of definition, a process justified by the principles of analytic continuation which allows us to uniquely continue a function defined over a region, but not over a set of isolated points (integers) [22]. It is significant that this procedure is independent of ω, reflecting the fact that the axial gauge singularity is spurious. The result (2.2) is a meromorphic function of ν although the original integral is singular if $A_2 \leqslant -(s+1)/2$. In (2.2) the G-function is singular whenever ν is a nonnegative integer, but this singularity is cancelled by the zero of $1/\Gamma(-\nu)$. So, S has no singularities when ν is an integer and the limit $\tau \to 0$ can be evaluated before all others—the spurious singularity has been regulated away. However, because ν is a continuous variable it is permissible to take derivatives with respect to ν in order to evaluate exponent derivatives [20]—integrals with integrands containing powers of $\ln(q \cdot n)^2$.

The regularization of the infrared and ultraviolet divergencies is somewhat more complicated, since these are end-point singularities and are therefore closely connected with the dimensionality of the integral. We regularize these divergencies, respectively, by initially choosing

$$\alpha_0 \geqslant -s/2,$$

$$\alpha_1 \geqslant -s/2,$$

and analytically continuing the result (2.2) in either ω (dimensional regularization) or μ and κ (analytic regularization) or both (hybrid).

In the method of dimensional regularization [1–4] one is limited to regulating the axial gauge singularity with the principle value prescription [12] (cf. Section 3). Insofar as the other singularities are concerned, one sets $\rho = \sigma = 0$ at the outset, performs analytic continuation in ω by letting $\varepsilon \to 0$ and identifies the terms of $O(1/\varepsilon)$ as the infinite parts of S. This method does not permit the computation of derivatives with respect to κ, μ and ν, nor the evaluation of integrals with M and K outside the limits given in (2.3b,c)—M and K must be negative integers in dimensional regularization.

In the method of analytic regularization [14], which must be invoked if exponent derivatives are desired, $\varepsilon = 0$ at the outset and the infinite parts of S appear as $O(1/\sigma)$ and/or $O(1/\rho)$ terms. As described earlier, the would-be axial gauge singularities of $O(1/\tau)$ do not appear.

In practice we choose the hybrid regularization which possesses the power of analytic regularization—it allows the simultaneous regularization of infrared, ultraviolet and axial gauge singularities—but retains the simplicity of dimensional regularization: allow all ε, ρ, σ and τ to be nonzero until after the S integral and/or exponent derivatives have been evaluated, then set $\rho = \sigma = \tau = 0$ and evaluate the limit $\varepsilon \to 0$.

A fundamental observation can now be made by inspecting the G-function in (2.6): all singularities of S due to divergencies of the original integral arise from singularities of the G-function—poles in the complex $(\omega, \kappa, \mu, \nu)$ space—that occur whenever the difference between one of the top three parameters and one of the first two bottom parameters is a positive integer. The fact that S depends on ε, ρ and σ through the indices of (2.5) assures that coefficients of the $O(1/\varepsilon)$ terms and those of the $O(1/\rho)$ and $O(1/\sigma)$ terms in S are the same, although those of higher-order ($O(1)$, $O(\varepsilon)$, $O(\rho)$, $O(\sigma)$, etc.) terms may differ. This is expected because there is no unique generalization of a function defined over a set of integers. For example, any regular function proportional to ρ, σ, τ or ε can be arbitrarily added to S with no effect on S, but with a profound effect on its exponent derivatives.

Finally, we consider the G-function in (2.2) as a function of y. The analytic continuation of a G-function outside the circle $|y| \leqslant 1$ is well defined, and in the case considered the result is another G-function valid for $|1/y| < 1$. In particular the point $1/y = 0^+$ corresponding to the case $n^2 = 0^+$ is accessible. This special case leads to representations useful in the light-cone gauge, to be discussed in Section 2.5.

2.3. *Contour Integral Representation for* $|y| < 1$

We may write the G-function in its contour integral representation [23],

$$S = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa+\nu}(n^2)^{\nu}\,\Gamma(s+\nu+\frac{1}{2})(p\cdot n)^s}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)\,\Gamma(2\nu+\mu+\kappa+s+2\omega)} \tag{2.7}$$

$$\times \frac{1}{2\pi i}\int_L dt\, y^t\, \frac{\Gamma(-t)\,\Gamma(\omega+\nu+\kappa-t)\,\Gamma(\mu+\nu+s+\omega+t)\,\Gamma(-\mu-\nu-\kappa-\omega+t)\,\Gamma(-\nu+t)}{\Gamma(\frac{1}{2}+s+t)},$$

where the contour L encloses the poles of the first two gamma functions and excludes the others. The situation is depicted schematically in Fig. 1. Note that inside the contour one string of poles of the integrand in (2.7) is fixed at the nonnegative integers whereas a second (β_1) string of poles approaches all but a finite number of the first only when ν and κ are integers and $\varepsilon \to 0$.

Exterior to the contour, we find "fixed" (independent of ε) poles of the integrand pinching the contour as $\tau \to 0$ and "moving" (ε-dependent) poles also pinching the contour as ρ, σ, τ and ε approach zero. Such pinches will be reflected as pole singularities of the contour integral at $\varepsilon = \rho = \sigma = \tau = 0$. In addition there exist both

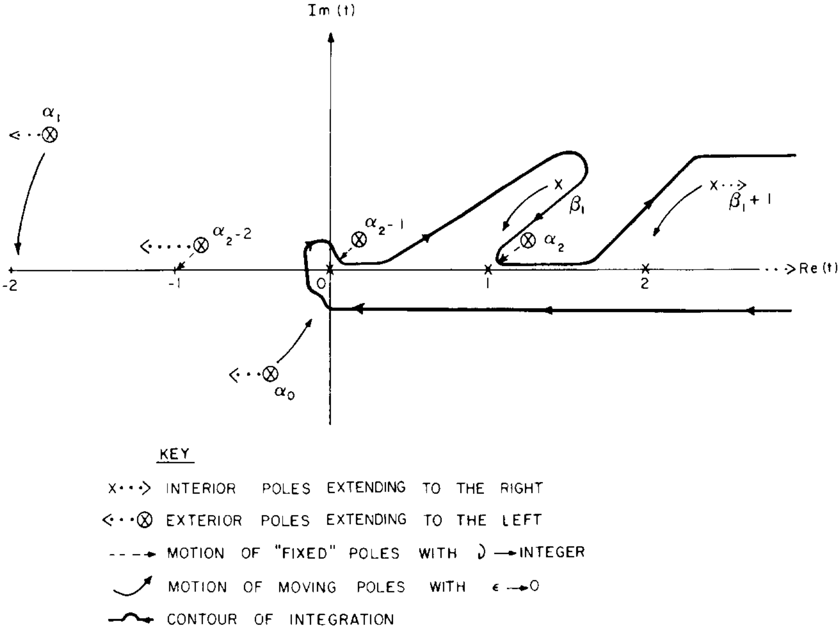

FIG. 1. Pole structure of the contour integral in (2.7) for the case $A_0 = 0$, $A_1 = -2$, $A_2 = 1$ and $B_1 = 1$, corresponding to $S(p, n; -2, -3, 1, 0)$ with $|y| < 1$.

fixed and moving zeros from the gamma functions exterior to the contour integral, acting to reduce the overall degree of singularity of S. The result is that S has simple poles at $\varepsilon = \rho = \sigma = 0$ in the ε, ρ, σ plane, verifying our earlier claim that S is free from axial gauge singularities and is regular at $\tau = 0$.

Alternatively, S may be viewed as a function of the indices of (2.5), as in (2.6). If we further define

$$\varepsilon_3 \equiv \varepsilon_b + \varepsilon_2, \tag{2.8}$$

then we find S has simple poles at $\varepsilon_i = 0$ in the ε_i-planes, $i = 0, 1, 3$. From Fig. 1, we observe that pinches in the contour have their genesis in three strings of exterior poles (α_0, α_1 and α_2) extending to the left and two strings of interior poles (one labelled β_1, the other being the nonnegative integers) extending to the right. A first kind of pinch singularity of the contour integral occurs whenever $\mathrm{Re}(\alpha_i) \geqslant 0$ ($i = 0, 1, 2$) and a second kind occurs whenever an α_i pinches β_1. The singularities generated by the α_2 (axial gauge singularity), $\alpha_0 - \beta_1$ and $\alpha_1 - \beta_1$ pinches are cancelled by corresponding zeros of the gamma functions exterior to the contour integral (see (2.6)). The three surviving singularities appear as poles of S and reflect the physical divergencies in the original integral (2.1)—infrared (α_0 and $\alpha_2 - \beta_1$) from singularities of the integral at $q^2 = 0$ and $q = p$, and ultraviolet (α_1).

By studying the interaction between the pinches and zeros as they coalesce, it is possible to demonstrate that the singularities of S are at most simple poles in the $(\omega, \kappa, \mu, \nu)$ space. The "overlap region" where the pinches reside contains a finite number of poles. Thus writing the integral in the form

$$S_{2\omega}(p, n) = \sum_{i=0,1,3} \frac{I_i(y, p, n)}{\varepsilon_i} + R(y, p, n), \tag{2.9}$$

where $I_i(y, p, n)$ are the numerators of the divergent (or infinite) part and $R(y, p, n)$ is the regular (or finite) part, we see that $I_i(y, p, n)$ is a function of y with a finite number of terms, since only pinches in the overlap region contribute to it. The regular part R may consist of an infinite series in y restricted to $|y| < 1$ due to poles of the integrand starting at $t > \mathrm{Max}(A_0, A_1, A_2, 0)$ and extending to $t \to +\infty$, plus a finite number of higher-order derivative terms surviving from the overlap region. I_i and R are also regular functions of the epsilons with leading $O(1)$ terms.

2.4. *Contour Integral Representation for* $|y| > 1$

From the theory of G-functions [24] it is possible to analytically continue the representation (2.2) into Minkowski space, i.e., the region $|y| > 1$. The result is

$$\begin{aligned} S = {} & \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa}(p \cdot n)^{2\nu+s}\,\Gamma(s+\nu+\frac{1}{2})}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)\,\Gamma(2\omega+2\nu+\mu+\kappa+s)} \\ & \times G^{3,2}_{3,3}\left(\frac{1}{y}\,\middle|\, \begin{matrix} 1+\nu, 1-\omega-\kappa; \frac{1}{2}+\nu+s \\ 0, -\omega-\mu-\kappa, \omega+\mu+2\nu+s; \end{matrix}\right) \end{aligned} \tag{2.10}$$

which has the contour integral representation

$$\begin{aligned} S = {} & \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa}(p \cdot n)^{2\nu+s}\,\Gamma(s+\nu+\frac{1}{2})}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)\,\Gamma(2\nu+\mu+\kappa+s+2\omega)} \\ & \times \frac{1}{2\pi i}\int_L dt\, y^{-t}\,\frac{\Gamma(-t)\,\Gamma(-\mu-\kappa-\omega-t)\,\Gamma(\mu+2\nu+s+\omega-t)\,\Gamma(-\nu+t)\,\Gamma(\kappa+\omega+t)}{\Gamma(\frac{1}{2}+\nu+s-t)} \end{aligned} \tag{2.11}$$

illustrated schematically in Fig. 2. The same comments hold as for (2.7) except that L now encloses three strings of interior poles extending to the right, and excludes two strings of moving and fixed exterior poles extending to the left. Again there is an overlap region pinching only a finite number of poles, so the contribution to the divergent terms $I_i(y, p, n)$ can at most contain a finite sum of $1/y$ terms.

Since finite sums are their own analytic continuation, it follows that all representations of $I_i(y, p, n)$ valid for $|y| < 1$ will be valid for $|y| > 1$; the same is true for the finite number of survivors from the overlap region that contribute to $R(y, p, n)$; these terms will contain factors of $1/y$ and $\ln y$. It is thus sufficient to evaluate the analytic continuation of any infinite series in $R(y, p, n)$ to obtain representations for $S_{2\omega}(p, n;$

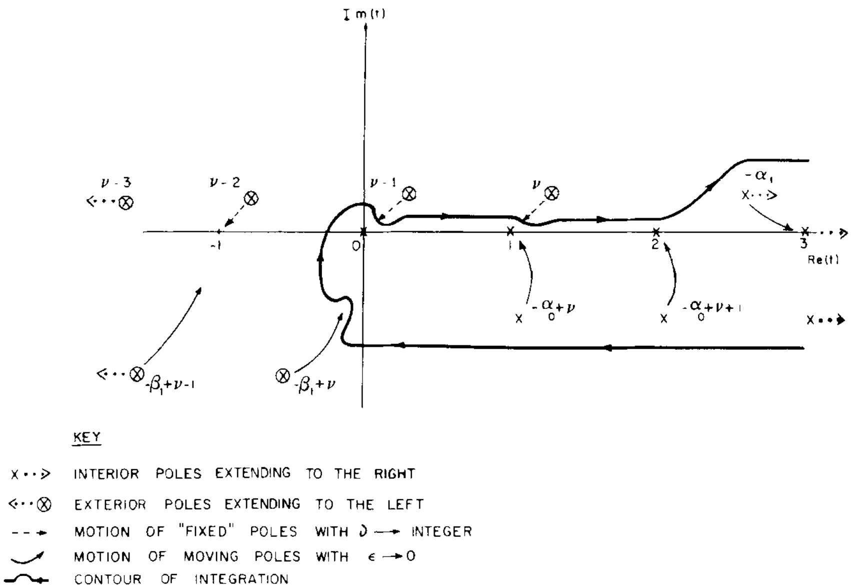

FIG. 2. Pole structure of the contour integral in (2.11) for the case $A_0 = 0$, $A_1 = 2$, $A_2 = 1$, $B_1 = 1$ corresponding to $S(p, n; -2, -3, 1, 0)$ with $|y| > 1$.

$\kappa, \mu, \nu, s)$ valid for all values of y. This is done explicitly by example in Section 4.1, and in general in Section 4.3.

2.5. *The Points $y = 0$ and $y = \infty$*

The special gauge, specified by the condition $p \cdot n = 0$ $(n^2 \neq 0)$ corresponding to the point $y = 0$, has recently become popular in nonperturbative studies of infrared properties of the gluon propagator [25]. The point $y = +\infty$ corresponds to the well-known light-cone gauge [26] specified by the condition $n^2 = 0$ $(p \cdot n \neq 0)$.

From (2.6) it is seen that the limit $y = 0^+$ is well defined in the region $\omega + \kappa + \nu > 0$ while the limit $y = +\infty$ is well defined in the region $-\mu - 2\nu - s < \omega < -\mu - \kappa$. These regions then provide bases for analytic continuation to obtain well-defined representations of the S-integral for the special $p \cdot n = 0$ and light-cone gauges. The results

$$S(p, n; \kappa, \mu, \nu, s)\,|_{p \cdot n = 0} = \delta_{s0} \frac{\pi^{\omega}(p^2)^{\omega+\kappa+\mu+\nu}(n^2)^{\nu}\, \Gamma(\nu+\frac{1}{2})\, \Gamma(\omega+\kappa+\nu)\, \Gamma(\omega+\mu+\nu)\, \Gamma(-\omega-\kappa-\mu-\nu)}{\Gamma(\frac{1}{2})\, \Gamma(-\kappa)\, \Gamma(-\mu)\, \Gamma(2\omega+\kappa+\mu+2\nu)}, \tag{2.12}$$

$$S(p, n; \kappa, \mu, \nu, s)\,|_{n^2 = 0} = \frac{\pi^{\omega}(p^2)^{\omega+\kappa+\mu}(p \cdot n)^{2\nu+s}\, \Gamma(\omega+\kappa)\, \Gamma(\omega+\mu+2\nu+s)\, \Gamma(-\omega-\kappa-\mu)}{\Gamma(-\kappa)\, \Gamma(-\mu)\, \Gamma(2\omega+\kappa+\mu+2\nu+s)} \tag{2.13}$$

are remarkably simple compared to the complexity of (2.2). Equation (2.12) was previously obtained by Alekseev [27].

Not surprisingly, due to the additional analytic continuation needed to obtain (2.12) and (2.13), the two special gauges have peculiar properties not shared by the general axial gauge. These properties are discussed in detail elsewhere [28].

3. Principal Value Prescription for the Axial Gauge Singularity

In the last section the axial gauge singularity, together with the ultraviolet and infrared divergencies, were analytically regularized; the regularized S-integral is proportional to a G-function. In this section we shall show that when the axial gauge singularity is treated with the principal value prescription, the resulting S-integral can be expressed as the limit of a polynomial in a differential operator operating on a series of G-functions, and furthermore, that this result is equivalent to the earlier one.

According to the principal value prescription [12] the axial gauge singularity ($N \geqslant 1$) is regularized by the limiting process

$$\left(\frac{1}{q\cdot n}\right)^{2N+s} \to \frac{1}{2}\lim_{\eta\to 0}\left[\left(\frac{1}{q\cdot n+i\eta}\right)^{2N+s}+\left(\frac{1}{q\cdot n-i\eta}\right)^{2N+s}\right] \tag{3.1a}$$

$$= \lim_{\eta\to 0}\left[\frac{1}{(q\cdot n)^2+\eta^2}\right]^{2N+s}\sum_{l=0}^{N}(q\cdot n)^{2N-2l+s}(-\eta^2)^l\binom{2N+s}{2l} \tag{3.1b}$$

$$= \lim_{\eta\to 0} R(\eta^2)\left[\frac{1}{(q\cdot n)^2+\eta^2}\right]^{N+s}(q\cdot n)^s, \tag{3.1c}$$

where $s=0$ or 1 and $R(\eta^2)$ is a differential operator in $\eta^2\partial/\partial\eta^2$ to be identified shortly. The regularization of the axial gauge singularity of an S-integral using this prescription thus involves the analysis of

$$S(p,n) = \lim_{\eta^2\to 0} R(\eta^2)\, T(p,n,\eta)$$

$$= \lim_{\eta^2\to 0} R(\eta^2)\int d^{2\omega}q(q^2)^\mu(q\cdot n)^s((p-q)^2)^\kappa((q\cdot n)^2+\eta^2)^\nu, \tag{3.2}$$

where ν can be taken to be an integer if desired.

The procedure of Appendix A is exactly applicable to the integral $T(p,n,\eta)$ save that the term in square brackets in (A.7) becomes

$$[\xi+\tau y(1-\xi)]^\zeta \to (D_y)^\zeta\left[1+\frac{\eta^2\tau}{(1-\tau)(1-\xi)\,D_y}\right]^\zeta \tag{3.3}$$

in the notation of (B.2) and (A.9). Rewrite the binomial in (3.3) as a contour integral, a technique first employed by Capper and Leibbrandt [29], and transpose with the double integral of (A.7). We find

$$T(p,n,\eta^2)=\frac{(\pi)^{\omega}(p^2n^2)^{\nu}(p\cdot n)^s(p^2)^{\omega+\mu+\kappa}}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)}$$
$$\times\frac{1}{2\pi i}\int_L dt\,\frac{\Gamma(-t)\,\Gamma(\nu+s+\frac{1}{2}-t)(\eta^2/p^2n^2)^t}{\Gamma(2\omega+\kappa+\mu+2\nu-2t+s)}$$
$$\times G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}t+1-s-\mu-\nu-\omega,\omega+1-t+\mu+\nu+\kappa,1+\nu-t;\\ 0,\omega+\kappa+\nu-t;\frac{1}{2}-s\end{matrix}\right) \tag{3.4}$$

using (B.5). The contour L stretches from $-i\infty$ to $+i\infty$ enclosing the poles of $\Gamma(\nu+1/2+s-t)$ on the right because of (B.6a). As $\eta^2\to 0$ this displays a series of singularities of the form $(\eta^2)^{\nu+1/2+s}$,... if $\mathrm{Re}(\nu)<-\frac{1}{2}-s$. These singularities may be removed by operating on T with

$$R(\eta^2)=\frac{\Gamma(\nu+\frac{1}{2}+s)}{\Gamma(\nu+N+\frac{1}{2}+s)}\left(s+\nu+\frac{1}{2}-\eta^2\frac{\partial}{\partial\eta^2}\right)\cdots\left(s+\nu+N-\frac{1}{2}-\eta^2\frac{\partial}{\partial\eta^2}\right), \tag{3.5}$$

where $N+s=[-\nu]$ ([ ] means greatest integer less than or equal to). This isolates the desired behaviour arising from the pole of $\Gamma(-t)$ at $t=0$. Now set $\nu=-N-s$ and identify

$$R(\eta^2)=\frac{\Gamma(-N+\frac{1}{2})}{\Gamma(\frac{1}{2})}\prod_{l=0}^{N-1}\left(-N+\frac{1}{2}+l-\eta^2\frac{\partial}{\partial\eta^2}\right). \tag{3.6}$$

That this operator is equal to the regulator of (3.1c) is verifiable by proving the consequential result

$$\frac{\Gamma(\frac{1}{2}-N)}{\Gamma(\frac{1}{2})}\prod_{l=0}^{N-1}\left(\frac{1}{2}-N+l-\eta^2\frac{\partial}{\partial\eta^2}\right)\frac{(q\cdot n)^s}{((q\cdot n)^2+\eta^2)^{N+s}}$$
$$=\left(\frac{q\cdot n}{(q\cdot n)^2+\eta^2}\right)^{2N+s}\sum_{l=0}^{N}(-\eta^2/(q\cdot n)^2)^l\binom{2N+s}{2l}. \tag{3.7}$$

Expand the factor $((q\cdot n)^2+\eta^2)^{-N-s}$ in the left-hand side of (3.7) as an infinite series in $\eta^2/(q\cdot n)^2$ to obtain

$$(q\cdot n)^s R(\eta^2)((q\cdot n)^2+\eta^2)^{-N-s} \tag{3.8a}$$
$$=(-)^N(q\cdot n)^{-2N-s}\frac{\Gamma(\frac{1}{2}-N)}{\Gamma(\frac{1}{2})}\sum_k\frac{\Gamma(N+s+k)\,\Gamma(N+\frac{1}{2}+k)(-\eta^2/(q\cdot n)^2)^k}{\Gamma(N+s)\,\Gamma(\frac{1}{2}+k)\,k!}$$
$$=\left(\frac{q\cdot n}{(q\cdot n)^2+\eta^2}\right)^{2N+s}{}_2F_1\left(-N,\frac{1}{2}-N-s;\frac{1}{2};-\eta^2/(q\cdot n)^2\right). \tag{3.8b}$$

The result (3.8b) is obtained by identifying the series in (3.8a) as a hypergeometric function, and applying Euler's transformation [30] to arrive at a truncated series. The right-hand side of (3.7) is easily cast into the form of (3.8b) whenever $s = 0$ or $s = 1$ by using the duplication and reflection properties of the gamma function.

With the operator R specified in (3.6), the regularization defined by (3.2) now reduces to the result (2.2) because of (3.4)—the two procedures are equivalent. The order of divergence in (3.4) as $\eta^2 \to 0$ is reflected in the pole structure of (2.2) as a function of ν—possible poles at negative integer values of $\nu + s + \frac{1}{2}$.

4. Some Examples

We evaluate some special cases to illustrate the power of the G-function representation and to illuminate some properties discussed generally in Section 2. Other examples are given elsewhere [20, 32] as are implications for renormalization and nonperturbation theory. The factor π^ω is isolated when S is singular, and is written as π^2 when S is regular.

4.1. *A Regular Integral*

The case $\mu = \kappa = \nu = -1$, $s = 1$ is of some interest because $\omega = 2$ lies within the window of convergence defined by (2.3); there is no analytic continuation except in y, nor any regularization. Consequently the integral will be well defined in ω, κ, μ and ν and analytic in y. From (2.2) we have

$$S = \frac{\pi^2}{p^2 n^2} \Gamma\left(\frac{1}{2}\right) (p \cdot n)\, G^{2,3}_{3,3}\left(y \,\middle|\, \begin{matrix} 0, 0, 0; \\ 0, 0; -\frac{1}{2} \end{matrix}\right) \tag{4.1}$$

and there is no necessity to take ε limits within the G-function because the contour is not pinched. From (2.8) and utilizing the residue theorem for a dipole we find

$$S = \frac{-\pi^2 (p \cdot n)\, \Gamma(\frac{1}{2})}{p^2 n^2} \sum_l \frac{\Gamma(1+l)}{\Gamma(\frac{3}{2}+l)} y^l \left[\psi(1+l) - \psi\left(\frac{3}{2}+l\right) + \ln y\right] \tag{4.2}$$

and the series converges for $|y| < 1$, albeit lackadaisically. An equivalent form for this integral has been given by van Neerven [31] (Appendix C).

The analytic continuation to $|y| > 1$ given in (2.12) leads to

$$\begin{aligned} S &= \frac{\pi^2 \Gamma(\frac{1}{2})}{p \cdot n} G^{3,2}_{3,3}\left(\frac{1}{y} \,\middle|\, \begin{matrix} 0, 0; \frac{1}{2} \\ 0, 0, 0; \end{matrix}\right) \\ &= \frac{\pi^2}{2p \cdot n} \sum_l \frac{(-y)^{-l}}{\Gamma(l+1)\,\Gamma(\frac{1}{2}-l)} \left\{ \left[\psi\left(\frac{1}{2}-l\right) - \psi(1+l) - \ln y\right]^2 \right. \\ &\quad \left. + 2\psi'\left(\frac{1}{2}\right) - \psi'(1+l) - \psi'\left(\frac{1}{2}-l\right) \right\}. \end{aligned} \tag{4.3}$$

The series slowly converges for $|y| > 1$; ψ and ψ' are polygamma functions (digamma and trigamma, respectively). As $1/y \to 0^+$, the leading behaviour is

$$S(p, n) \sim \ln^2 y, \tag{4.4}$$

demonstrating that the function $R(y, p, n)$ of (2.10) is singular at the point $n^2 = 0^+$, although the leading pole structure in Fig. 2 and the condition (2.13) for this example nominally imply $(1/y^0)$ behaviour. Thus special treatment will be required for this integral in the light-cone gauge [28].

Explicit evaluations of all integrals with $\kappa = -1$, M and N integer are given in Ref. [32], and for a larger set of parameters in Ref. [40].

4.2. *Covariant Gauge*

Here we are interested in the case in which $\nu = s = 0$; S reduces to

$$S(p; \kappa, \mu) \equiv \int d^{2\omega}q (q^2)^{\mu} [(p-q)^2]^{\kappa} \tag{4.5}$$

and all n-dependence vanishes. From contiguity relations between G-functions [21], it is easy to extract a factor $1/\nu$ from the G-function in (2.2), write the sum of contiguous G-functions as a contour integral [23] and set $\nu = 0$. We find that all residues vanish save the one at the origin and no limiting process is required. The result is [4]

$$S(p; \kappa, \mu) = \frac{\pi^{\omega} (p^2)^{\omega + \mu + \kappa}\, \Gamma(\omega + \kappa)\, \Gamma(\omega + \mu)\, \Gamma(-\omega - \mu - \kappa)}{\Gamma(-\mu)\, \Gamma(-\kappa)\, \Gamma(2\omega + \mu + \kappa)}. \tag{4.6}$$

Note the symmetry with respect to interchange of μ and κ, as expected from the "shift" property of the integral representation (4.5). In addition, because of the order of limits, it is obvious from (4.6) that

$$S(p; K, \mu) = 0, \qquad K \geqslant 0, \tag{4.7a}$$

$$S(p; \kappa, M) = 0, \qquad M \geqslant 0. \tag{4.7b}$$

This confirms a conjecture relating to the properties of tadpole diagrams [33],

$$S(p; 0, M) = 0, \qquad M \geqslant 0. \tag{4.8}$$

Because of the conditions (2.3b,c) and reasons given in Section 2.2, (4.7) and (4.8) cannot be derived with the method of dimensional regularization alone. See Ref. [4] for a more detailed discourse on this subject.

4.3. *The General Case*

First of all, we note the generalizations of (4.7):

$$S(p, n; K, \mu, \nu, s) = 0, \qquad K \geqslant 0,$$
$$S(p, n; \kappa, M, N, s) = 0, \qquad M \text{ and } N \geqslant 0, \tag{4.9}$$

which are results unobtainable in dimensional regularization [13].

Because of the limited number of possible permutations in the overlap region (see Section 2.3), it is feasible to classify any S-integral according to the arrangement of the poles and zeros. We expand to first order in ε, and introduce an encompassing notation. Write

$$S(p, n) = \frac{T}{D} \mathscr{G}, \tag{4.10a}$$

where $\mathscr{G}$ is given in Table I,

$$T = \pi^{\omega} (p^2)^{2+M+K+N} (n^2)^N (p \cdot n)^s \Gamma(\tfrac{1}{2} + N + s)$$
$$\times [1 + \varepsilon(\ln p^2 - 2\bar{\psi}(B_1 - A_0) - 0\bar{\psi}(B_1 - A_1))], \tag{4.10b}$$

$$D = \bar{\Gamma}(B_1 - A_0)\, \bar{\Gamma}(B_1 - A_1)\, \bar{\Gamma}(-A_2)\, \bar{\Gamma}(-A_0 - A_1 - s) \tag{4.10c}$$

and

$$\bar{\Gamma}(x) = \Gamma(x), \qquad \text{if} \quad x > 0,$$
$$= (-)^x / \Gamma(1 - x), \qquad \text{if} \quad x \leqslant 0, \tag{4.10d}$$

$$\bar{\psi}(x) = \psi(x), \qquad \text{if} \quad x > 0,$$
$$= \psi(1 - x), \qquad \text{if} \quad x \leqslant 0. \tag{4.10e}$$

The parameters A_0, A_1, A_2 and B_1 are given in (2.5), and the "0" and "2" coefficients are discussed in the footnote to Table I. The function $g^{(i)}(b \mid a)$ appearing in Table I is defined by

$$g^{(i)}(b \mid a) = \sum_{l=0}^{b-a} g(a, l)\, \Psi_i(l + a)\, y^l, \tag{4.11a}$$

where

$$g(a, l) = (-y)^a \frac{\bar{\Gamma}(B_1 - a) \prod_{i=0}^{2} \bar{\Gamma}(a - A_i)(a - A_i)_l}{\bar{\Gamma}(a + 1)\, \Gamma(a + \frac{1}{2} + s)(a + \frac{1}{2} + s)_l (a - B_1 + 1)_l (a + 1)_l} \tag{4.11b}$$

with

$$\Psi_0(l) = 1, \tag{4.11c}$$

$$\Psi_1(l) = 1 + \varepsilon\bar{\psi}(B_1 - l) + \varepsilon_0 \bar{\psi}(-A_0 + l) + \varepsilon_1 \bar{\psi}(-A_1 + l), \tag{4.11d}$$

$$\Psi_2(l) = -1 - \varepsilon(\ln y - \bar{\psi}(-l) + \bar{\psi}(-A_2 + l) - \psi(\tfrac{1}{2} + s + l)$$
$$+ 2\bar{\psi}(-A_0 + l) + 0\bar{\psi}(-A_1 + l)), \tag{4.11e}$$

TABLE I

Evaluation of $\mathscr{S}$ in Eq. (4.10a)

Case	Condition	Infinite part[a]	Regular part
		Class A[b]: $A_2 < 0$	
A.1	$B_1 = 0 > A_0 \geqslant A_1$	—	$Z(A_0, A_1, B_1 - 1)$
A.2	$A_0 \geqslant B_1 > 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0 \mid B_1)$	$2g^{(0)}(B_1 - 1 \mid 0) + g^{(3)}(A_0 \mid B_1)$
A.3	$A_0 \geqslant 0 \geqslant B_1 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0 \mid 0)$	$-g^{(0)}(-1 \mid B_1) + g^{(3)}(A_0 \mid 0)$
A.4	$A_0 \geqslant 0 > A_1 \geqslant B_1$	—	$-g^{(0)}(A_1 \mid B_1) - 0g^{(0)}(A_0 \mid 0)$
A.5	$0 > A_0 \geqslant A_1 \geqslant B_1$	—	$-g^{(0)}(A_1 \mid B_1)$
A.6	$A_0 \geqslant 0 > A_1 > A_2 \geqslant B_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_2 \mid B_1)$	$g^{(0)}(A_1 \mid A_2 + 1) - 0g^{(0)}(A_0 \mid 0)$
A.7	$A_0 \geqslant 0 > A_2 \geqslant B_1 > A_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_2 \mid B_1) + \frac{1}{\varepsilon_0} g^{(1)}(A_0 \mid 0)$	$-g^{(0)}(-1 \mid A_2 + 1) + g^{(3)}(A_0 \mid 0)$
A.8	$A_0 \geqslant 0 > A_2 \geqslant A_1 \geqslant B_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_1 \mid B_1)$	$-0g^{(0)}(A_2 \mid A_1 + 1) - 0g^{(0)}(A_0 \mid 0)$
A.9	$0 > A_0 \geqslant A_1 \geqslant A_2 \geqslant B_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_2 \mid B_1)$	$-g^{(0)}(A_1 \mid A_2 + 1)$
A.10	$0 > A_0 \geqslant A_2 > A_1 \geqslant B_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_1 \mid B_1)$	$0g^{(0)}(A_2 \mid A_1 + 1)$
A.11	$0 > A_2 \geqslant A_0 \geqslant B_1 > A_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_0 \mid B_1)$	$-g^{(0)}(-1 \mid A_2 + 1) - 2g^{(0)}(A_2 \mid A_0 + 1)$
A.12	$0 > A_2 > A_0 > A_1 \geqslant B_1$	$\frac{1}{\varepsilon_3} g^{(2)}(A_1 \mid B_1)$	$-0g^{(0)}(A_0 \mid A_1 + 1)$
		Class B[b]: $A_2 \geqslant 0$	
B.1	$A_2 \geqslant B_1 \geqslant 0 > A_0 \geqslant A_1$	$\frac{1}{\varepsilon_3} g^{(1)}(A_2 \mid B_1)$	$g^{(0)}(B_1 - 1 \mid 0)$

Note. Symbols and notation: for A_i, B_1 and ε_i see (2.5); $\varepsilon_3 \equiv \varepsilon_b + \varepsilon_2$; for $g^{(i)}(a \mid b)$ see (4.11), for Z see (4.12). In the limit $\rho = \sigma = \tau = 0$, $\varepsilon_0 = \varepsilon_3 = \varepsilon$ and $\varepsilon_1 = -\varepsilon$. $\mathscr{S}$ is the sum of infinite and regular parts.

[a] The "infinite" part given here includes terms of $O(1/\varepsilon)$ as well as those $O(1)$ terms that are naturally associated with the $O(1/\varepsilon)$ terms.

[b] Class A cases all have $A_2 < 0$, $A_0 \geqslant A_1$ and $B_1 > A_2$ if the position of A_2 is not given; if $A_0 + A_1 + s \geqslant 0$, then $\mathscr{S} = 0$. Class B cases all have $A_2 \geqslant 0$, $A_0 \geqslant A_1$ and $B_1 > A_1$; if $A_1 \geqslant B_1$ then $\mathscr{S} = 0$. Corresponding expressions for $A_1 > A_0$ are obtained by interchanging A_0 and A_1 and the coefficients 2 and 0, and replacing ε_0 by ε_1, in the table and in (4.10) and (4.11).

Table continued

TABLE I (*continued*)

Case	Condition	Infinite part[a]	Regular part
B.2	$A_2 \geqslant B_1 > A_0 \geqslant 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0 \mid 0) + \frac{1}{\varepsilon_3} g^{(1)}(A_2 \mid B_1)$	$g^{(0)}(B_1 - 1 \mid A_0 + 1)$
B.3	$A_2 > A_0 \geqslant B_1 \geqslant 0 > A_1$	$\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}\right) g^{(1)}(A_0 \mid B_1)$	$2g^{(0)}(B_1 - 1 \mid 0) + 2g^{(0)}(A_2 \mid A_0 + 1)$
B.4	$A_2 > A_0 \geqslant 0 > B_1 > A_1$	$\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}\right) g^{(1)}(A_0 \mid 0)$	$2g^{(0)}(A_2 \mid A_0 + 1)$
B.5	$A_2 \geqslant 0 > B_1 > A_0 \geqslant A_1$	$\frac{1}{\varepsilon_3} g^{(1)}(A_2 \mid 0)$	—
B.6	$A_2 \geqslant 0 > A_0 \geqslant B_1 > A_1$	—	$2g^{(0)}(A_2 \mid 0)$
B.7	$B_1 > A_2 \geqslant 0 > A_0 \geqslant A_1$	—	$g^{(0)}(A_2 \mid 0)$
B.8	$B_1 > A_2 > A_0 \geqslant 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0 \mid 0)$	$g^{(0)}(A_2 \mid A_0 + 1)$
B.9	$B_1 > A_0 \geqslant A_2 \geqslant 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_2 \mid 0)$	—
B.10	$A_0 \geqslant A_2 \geqslant B_1 \geqslant 0 > A_1$	$\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}\right) g^{(1)}(A_2 \mid B_1)$	$2g^{(0)}(B_1 - 1 \mid 0)$
B.11	$A_0 \geqslant A_2 \geqslant 0 > B_1 > A_1$	$\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}\right) g^{(1)}(A_2 \mid 0)$	—
B.12	$A_0 \geqslant B_1 > A_2 \geqslant 0 > A_1$	—	$2g^{(0)}(A_2 \mid 0)$

$$
\begin{aligned}
\Psi_3(l) &= (\Psi_1(l) + \Psi_2(l))/\varepsilon \\
&= \bar{\psi}(B_1 - l) + \bar{\psi}(-l) - \bar{\psi}(-A_0 + l) - \bar{\psi}(-A_1 + l) \\
&\quad - \bar{\psi}(-A_2 + l) + \psi(\tfrac{1}{2} + s + l) - \ln y,
\end{aligned} \tag{4.11f}
$$

and the convention that $g^i(b \mid a) = 0$ if $b < a$; $(\ )_l$ is Pochhammer's symbol. The function Z in Table I is defined by

$$
\begin{aligned}
Z(A_0, A_1, B_1 - 1) &= \sum_l g(a, l)\, \Psi_3(l + a)\, y^l, & &\text{if } |y| \leqslant 1, \\
&= \sum_l \tilde{g}(l + d)\, \Psi_4(l + d)(-y)^{-l-d}, & &\text{if } |y| > 1,
\end{aligned} \tag{4.12a}
$$

where

$$
a = \mathrm{Max}(0, B_1, A_0 + 1, A_1 + 1), \tag{4.12b}
$$

$$
d = \mathrm{Max}(0, 1 - B_1 + A_2, 1 + A_2), \tag{4.12c}
$$

with

$$\tilde{g}(l) = (\tfrac{1}{2})(-)^{1-A_0-A_1} \times \frac{\Gamma(B_1 - A_2 + l)\,\Gamma(-A_2 + l)}{\Gamma(1+l)\,\Gamma(1 + A_1 - A_2 + l)\,\Gamma(1 + A_0 - A_2 + l)\,\Gamma(\tfrac{1}{2} + A_2 + s - l)} \tag{4.12d}$$

and

$$\begin{aligned}\Psi_4(l) = {} & [\psi(B_1 - A_2 + l) + \psi(l - A_2) - \psi(1+l) - \psi(1 + A_1 - A_2 + l) \\ & - \psi(1 + A_0 - A_2 + l) + \psi(\tfrac{1}{2} + A_2 + s - l) + \ln(1/y)]^2 \\ & + \pi^2 + \psi'(B_1 - A_2 + l) + \psi'(l - A_2) - \psi'(1+l) \\ & - \psi'(1 + A_1 - A_2 + l) - \psi'(1 + A_0 - A_2 + l) - \psi'(\tfrac{1}{2} + A_2 + s - l).\end{aligned} \tag{4.12e}$$

Nonzero values for the functions in (4.10a) are given in Table I. Although it is possible to extend Table I to higher orders in ρ, τ and σ in order to calculate exponent derivatives, we refrain from doing so because the general expressions rapidly become unmanageably lengthy and complicated. However, the general evaluation may be easily executed by computer (SCHOONSCHIP [34]) or by hand with some practice. Most frequently encountered cases are explicitly tabulated in Ref. [40].

5. Summary

In this paper we have presented a well-defined method of evaluating a class of divergent integrals that arise in quantum field theory. All possible integrals belonging to this class are summarized in Table I, and are easily evaluated [32]. For the axial gauge, compared to the commonly used principal value prescription, the new method embodies a simple, elegant, well-defined and, most important, mechanical technique. At the same time, we have demonstrated that the new method generates the same results as the older prescription. Thus, all results will retain the fundamental properties deduced for the older methods—in particular, gauge invariance. This is demonstrated explicitly elsewhere [18, 28].

Because of the inherent simplicity of our method, we are able to obtain a new result for the light-cone gauge, yet reproduce known results for other, simpler gauges. The new method also naturally isolates infrared from ultraviolet singularities [41].

Appendix A

In this and Appendix B, the main result (2.2) will be derived. We begin by replacing three of the factors in (2.1) by an integral representation of the form

$$(q^2)^\mu = \frac{1}{\Gamma(-\mu)} \int_0^\infty t^{-\mu-1} e^{-q^2 t}\, dt, \qquad \mathrm{Re}(\mu) < 0, \tag{A.1}$$

and transpose with the outermost integral to obtain

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \frac{1}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)} \int_0^\infty dt \int_0^\infty du \int_0^\infty dv\, t^{-\mu-1} u^{-\nu-1} v^{-\kappa-1} e^{-p^2 v} J(t, u, v),$$

$$\operatorname{Re}(\mu, \nu, \kappa) < 0, \quad \text{(A.2)}$$

where

$$J(t, u, v) = \int d^{2\omega} q (q \cdot n)^s \exp(-q^2(t+v) + 2(p \cdot q)\, v - (q \cdot n)^2\, u). \quad \text{(A.3)}$$

The above integral has been derived elsewhere [13]; it is equivalent to

$$\int d^{2\omega} q (q \cdot n)^s\, e^{-\alpha q^2 + 2\beta p \cdot q - \gamma (q \cdot n)^2} = \left(\frac{\pi}{\alpha}\right)^\omega \left(\frac{\beta p \cdot n}{\alpha + \gamma n^2}\right)^s \frac{\alpha^{1/2}}{(\alpha + \gamma n^2)^{1/2}} \exp\left(\left[\beta^2 p^2 - \frac{\gamma \beta^2 (p \cdot n)^2}{\alpha + \gamma n^2}\right] \Big/ \alpha\right) \quad \text{(A.4)}$$

which has the feature that it reproduces the usual results whenever ω is a positive half-integer. In point of fact, we may always take ω equal to a positive half-integer in (A.3) and (A.4), provided μ, ν and κ are continuous variables, as follows from the discussion in Section 2. Thus (A.2) becomes

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \frac{\pi^\omega (p \cdot n)^s}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)} \int_0^\infty dt \int_0^\infty du \int_0^\infty dv\, \frac{t^{-\mu-1} u^{-\nu-1} v^{-\kappa-1+s} (t+v)^{1/2-\omega}}{(t + v + un^2)^{1/2+s}}$$
$$\times \exp\left[\frac{p^2 v^2}{t+v} - \frac{u v^2 (p \cdot n)^2}{(t+v)(t+v+un^2)} - p^2 v\right]. \quad \text{(A.5)}$$

Now transform the variables according to

$$t = (1-\tau)\,\xi\lambda,$$
$$u = \lambda\tau/n^2, \quad \text{(A.6)}$$
$$v = \lambda(1-\tau)(1-\xi),$$

and the scale integration (λ, from 0 to ∞) may be evaluated analytically using (A.1). The eventual result is

$$S_{2\omega}(p, n; \kappa, \mu, \nu, s) = \frac{\pi^{\omega}(n^2)^{\nu}(p \cdot n)^s (p^2)^{\omega+\mu+\kappa+\nu}\,\Gamma(-\omega-\mu-\nu-\kappa)}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)}$$
$$\times \int_0^1 d\tau \int_0^1 d\xi\, \tau^{-\nu-1}(1-\tau)^{-1/2+\nu+s}\,\xi^{-\mu-1}(1-\xi)^{\omega+\mu+\nu+s-1}$$
$$\times\,[\xi + \tau y(1-\xi)]^{\omega+\mu+\nu+\kappa}, \tag{A.7}$$

with

$$y = (p \cdot n)^2/(p^2 n^2),$$

if

$$\mathrm{Re}(\omega + \mu + \nu + \kappa) < 0. \tag{A.8}$$

The double integral in (A.7) is in the canonical form of an integral evaluated in Appendix B, with the substitutions

$$\begin{aligned} &\alpha \to -(\omega + \kappa + \mu + 2\nu + 1),\\ &\beta \to -\tfrac{1}{2} + \nu + s,\\ &\gamma \to \omega + \mu + \nu + s - 1,\\ &\mu \to \mu + 1,\\ &\zeta \to \omega + \mu + \nu + \kappa. \end{aligned} \tag{A.9}$$

Comparing with our approach, Bollini *et al.* [14] use a generalized Feynman formula instead of the generalized exponentiation of (A.1). Like us, they use analytic continuation to define divergent integrals. Speer [14] regulates propagators (as opposed to only integrals), uses (A.1) in a modified form—the lower limit of integration is replaced by r and the limit $r \to 0^+$ is considered—and does not explicitly use the principle of analytic continuation. While Speer's method may not preserve gauge invariance, the fact that our method preserves gauge invariance is demonstrated explicitly elsewhere [18, 41].

Appendix B

We wish to evaluate the double integral

$$\mathscr{D} = \int_0^1 d\xi \int_0^1 d\tau\, \tau^{\alpha+\zeta}(1-\tau)^{\beta}(1-\xi)^{\gamma}\,\xi^{-\mu} D_y^{\zeta}, \tag{B.1}$$

where

$$D_y = \xi + (1-\xi)\,\tau y. \tag{B.2}$$

Perform the transformation $v = (1-\xi)/\xi$, and recognizing the τ integral as the integral representation of a hypergeometric function [35] obtain

$$\mathcal{L} = \frac{\Gamma(\alpha+\zeta+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+\zeta+2)} \int_0^\infty dv\, v^\gamma (1+v)^{\mu-2-\gamma-\zeta} \times {}_2F_1(-\zeta, \alpha+\zeta+1; \alpha+\beta+\zeta+2; -vy). \tag{B.3}$$

Express the hypergeometric function as a G-function [36] so that

$$\mathcal{L} = \frac{\Gamma(\beta+1)}{\Gamma(-\zeta)} \int_0^\infty dv\, v^\gamma (1+v)^{\mu-2-\gamma-\zeta}\, G_{2,2}^{1,2}\left(vy \,\middle|\, \begin{matrix} 1+\zeta, -\alpha-\zeta; \\ 0; -1-\alpha-\beta-\zeta \end{matrix}\right). \tag{B.4}$$

This integral is known [37], and the final result is

$$\mathcal{L} = \frac{\Gamma(\beta+1)}{\Gamma(-\zeta)\,\Gamma(2+\gamma+\zeta-\mu)}\, G_{3,3}^{2,3}\left(y \,\middle|\, \begin{matrix} -\gamma, 1+\zeta, -\alpha-\zeta; \\ 0, 1+\zeta-\mu; -1-\alpha-\beta-\zeta \end{matrix}\right), \tag{B.5}$$

if

$$\mathrm{Re}(2+2\zeta+\alpha-\mu) > 0, \tag{B.6a}$$

$$\mathrm{Re}(\gamma) > -1, \tag{B.6b}$$

$$\mathrm{Re}(\mu) < 1, \tag{B.6c}$$

$$\mathrm{Re}(1+\beta) > 0, \tag{B.6d}$$

$$\mathrm{Re}(\alpha+\zeta) > -1. \tag{B.6e}$$

The conditions (B.6a–e) may be relaxed on the right-hand side of (B.5), since the G-function is well defined for all values of its parameters.

Appendix C

We wish to show the equivalence of van Neerven's representation [31] for $I_{111} = (i/16\pi^4)\, S(p, n; -1, -1, -1, 1)$ and our (4.1). According to van Neerven,

$$I_{111} = \frac{i}{16\pi^2} \frac{1}{p \cdot n} F(x), \tag{C.1}$$

where

$$F(x) = 2\int_0^1 dt \frac{1}{1+t^2(x-1)} \ln\left(\frac{1+t}{1-t}\right), \qquad 1 < x < \infty. \tag{C.2}$$

Identify the logarithmic term with a well-known power series in hypergeometric form [38] and obtain

$$F(x) = 2\int_0^1 dv\,(1-(1-x)\,v)^{-1}\,{}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; v\right) \qquad \text{(C.3)}$$

after an obvious transformation of variables. Now impose a linear transformation $(v \to v/(v-1))$ on the hypergeometric function and again transform the variables $(t = v/(1-v))$, obtaining

$$F(x) = 2\int_0^\infty dt\,(1+xt)^{-1}\,{}_2F_1\left(1, 1; \frac{3}{2}; -t\right). \qquad \text{(C.4)}$$

Write the hypergeometric function as a G-function [36], and use a known integration formula [37]. Thus

$$F(x) = \frac{\Gamma(\frac{1}{2})}{x}\,G^{2,3}_{3,3}\left(\frac{1}{x}\,\middle|\,\begin{matrix}0,0,0;\\ 0,0;-1/2\end{matrix}\right), \qquad \frac{1}{x} < 1, \qquad \text{(C.5)}$$

demonstrating that

$$I_{111} = i\,\frac{\Gamma(\frac{1}{2})\,p\cdot n}{16\pi^2 p^2 n^2}\,G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}0,0,0;\\ 0,0;-1/2\end{matrix}\right), \qquad y \leqslant 1, \qquad \text{(C.6)}$$

using $y = 1/x$. This is precisely (4.1), after allowing for the different normalization. By the principles of analytic continuation, both representations are valid for all values of y.

Acknowledgments

We thank Graham Lee-Whiting for discussions on the regulator for the principal value prescription. We are especially thankful to George Leibbrandt for many informative discussions and communications.

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