# AN ANALYTIC REGULARIZATION FOR SUPERSYMMETRY ANOMALIES

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## 1. INTRODUCTION TO ANALYTIC REGULARIZATION

In the originally devised form of dimensional regularization [1-4], a theory in D (= integer) dimensions is defined as the limit $\varepsilon \to 0$ of a theory in $d \equiv D + 2\varepsilon$ dimensions. The method is incompatible with supersymmetry [5-8], since the algebras for vectors and spinors in a supersymmetric theory depend differently on D, so that supersymmetry is manifestly broken in the method.

In a modification, known as dimensional reduction [9], the algebra of Dirac matrices (which act only on spinors) is kept in the N-dimensional space, but the Lorentz algebra lives in d-dimensional space. This approach is basically inconsistent [9,10], but except for anomalies it can be made with special provisions to work for most purposes.

Dimensional regularization also fails when used to compute axial anomalies [11-15] since a totally antisymmetric tensor cannot be realized in a nonintegral-dimensional space. There is another more technical reason. Because anomalies (in perturbation theory) are closely related to surface integrals [17-19] - not surprising since in the topological approach [15] anomalies are associated with boundaires of manifolds - they are always expressible as finite differences of pairs of divergent Feynman integrals. But the nature of analytic continuation employed by dimensional regularization is such that these differences are exact cancellations so that the method always (incorrectly) gives zero values for anomalies.

Several modifications have been devised for dimensional regularization for the purpose of making it useful for the computing anaomalies [1,20-23].

The modification usually centres on the properties of the totally antisymmetric tensor and Dirac matrix ($\gamma_5$ when D=4) and requires special care for its application, thus tarnishing the elegance of the method in some way.

Another misconception that gives doubts to the applicability of dimensional regularization in computing anomalies is the belief that shift of integration variables in Feynman integrals is not allowed (a notion not inconsistent with the association between anomalies and surface integrals). In fact shift of integration variables is permitted, provided it is done consistently (in a way explained later). This is important because the shift operation is essential for an analytic approach.

Our goal is to find a regularization method based on analytic continuation that (i) preserves supersymmetry and (ii) is useful for computing anomalies. It goes without saying that the method must also preserve gauge invariance, Lorentz invariance, and so on. The method, which shall be called analytic regularization for reasons that will become obvious, is based on the observations that: (i) In (the perturbative approach to) field theory divergences that need to be regulated occur only in Feynman integrals; (ii) All amplitudes involving Feynman integrals and transforming as tensor of the Lorentz group can be expressed as sums of terms that are products of known Lorentz tensors and Lorentz invariant Feynman integrals. Thus in the proposed method only (Lorentz) invariant (Feynman) integral are regulated.

A method of regulating only invariant integrals entails a retreat from the more ambitious "global" approach of dimensional regularization, in which a theory is regulated. As mentioned earlier, the retreat is necessary if the objective is to preserve supersymmetry.

There is an earlier (and different) version of analytic regularization [24-26] often referred to as Speer's method. This approach is also global: propagators having the form $(k^2 + m_i{}^2)^{-1}$ is replaced by the expression $(k^2 + m_i{}^2)^{-\lambda_i}$, and the original theory is viewed as the limit $\lambda_i \to 1$ of the theory with the new propagators. In principle the method calls for as many parameters $\lambda_i$ as there are species of particles, a procedure which manifestly breaks supersymmetry. The method has had some success in verifying supersymmetric Ward identities at the one-loop level, provided the different $\lambda_i$'s are identified as a single parameter [27].

As a vehicle for testing our method, we consider the N=1 supersymmetric Yang-Mills theory [5-8] in the Wess-Zumino gauge, and use the method to verify the anomalous supersymmetric identity [28-35]

$$\delta(\mathcal{S}\cdot\gamma) = 2i\bar{\varepsilon}\ \mathrm{Tr}(\mathcal{T}) + 3\bar{\varepsilon}\ \gamma_5(\partial\mathcal{J}_5) \tag{1}$$

at the one-loop order. In the equations $\delta$ denotes the supersymmetric transformation, $\varepsilon$ is a constant spinor field, $\mathcal{S}$ is the supersymmetry current, $\mathcal{T}$ is the energy-momentum tensor and $\mathcal{J}_5$ is the axial-vector current. The non-vanishing quantities $\mathcal{S}\cdot\gamma$, $\mathrm{Tr}(\mathcal{T})$ and $\partial\mathcal{J}_5$ are anomalous since they respectively signal the violations of conformal invariance, energy-momentum conservation and chiral invariance due to quantum fluxuation. Eq. (1) is therefore an unbroken supersymmetric relation among anomalies. It will be shown that our method neatly separates the task of evaluating the three anomalies and the task of making manifest the supersymmetric relations connecting them.

## 2. N=1 SUPERSYMMETRIC YANG-MILLS THEORY [5-8]

We restrict ourselves to working in the Wess-Zumino gauge [5]. Then the Lagrangian is exists, which we call $M_1$. We now define

$$I_1(D,\ell_1,\ell_2,\ell_3) \overset{\text{def}}{=} \lim_{\substack{2\omega\to D\\ \lambda_i\to\ell_i}} M_1(\omega,\lambda_1,\lambda_2,\lambda_3) \tag{29}$$

The other two classes of integrals $I_2$ and $I_3$ are similarly defined. Because (we assume, and it turns out to be so) $M_1$ is an analytic function of $\omega$ and the $\lambda_i$'s, divergences in the original integral $I_1$ will appear in $M_1$ as poles in the $\omega\lambda$-plane. The method differs from dimensional regularization through the presence of generalized exponents $\lambda_i$. Since there are more than one parameter, the regularization is not completely defined until the limiting process in (29) is specified. We defer the choosing of the limit to the end of the computation.

Elsewhere [36,37] it has been shown that the multi-parameter continuation has allowed our method to render a rigorous treatment of certain integrals (e.g. tadpoles) undefinable in dimensional regularization and to analytically separate ultraviolet from infrared and mass singularities.

With the on-shell conditions (23) and (24), and using Euler's representation for exponentiating the factors in the integrand in (28) and standard dimensional regularizations technique [1,4,38] for the k-integration one derives [39] the close-form representation for $M_1$

$$M_1(\omega,\lambda_1,\lambda_2,\lambda_3) = \frac{(-\pi)^{\omega}(p^2)^{\alpha_1}\Gamma(-\alpha_1)\Gamma(\alpha_1-\lambda_2)\Gamma(\alpha_1-\lambda_3)}{\Gamma(-\lambda_2)\Gamma(-\lambda_3)\Gamma(2\alpha_1-\lambda_1-\lambda_2-\lambda_3)} \tag{30}$$

where $\alpha_1 \equiv \omega+\lambda_1+\lambda_2+\lambda_3$. Similar calculation shows that both $M_2$ and $M_3$ are identical to the right-hand side of (30). We therefore suppress the subscript on M henceforth. Now write

$$\omega = D/2 + \varepsilon, \qquad \lambda_i = \ell_i + \sigma_i \tag{31}$$

with the small continuous parameter $\varepsilon$ and the $\sigma_i$'s being of the same order. Then

$$I_{1,2,3}(D,\ell_1,\ell_2,\ell_3) \overset{\text{def}}{=} \lim_{\varepsilon,\sigma_i\to 0} M(\tfrac{D}{2}+\varepsilon,\ \ell_1+\sigma_1,\ \ell_2+\sigma_2,\ \ell_3+\sigma_3) \tag{32}$$

where, as usual, the limit $\varepsilon\to 0$ means ignoring terms of $\mathcal{O}(\varepsilon)$ and higher. To complete the regularization we still need to specify the approach to the origin in the $\varepsilon\sigma_i$-hyperplane.

## 4. PRESERVATION OF SUPERSYMMETRY

The method described in the last section clearly preserves Lorentz invariance and leaves unchanged the Dirac algebra. It does not manifestly break gauge invariance or supersymmetry, but neither is it guaranteed that such symmetries are preserved. Since properties of such symmetries are not transparently transmitted to Feynman integrals, it is difficult to directly assess whether our method indeed preserves them. Normally Ward identities and identities such as (1) are used for the purpose of making such assessments. This we shall do in the next section. Here we show how we can, by constraining the small parameters $\varepsilon$, $\sigma_i$ expect our regularization to preserve the symmetries.

We begin with the supposition that if the representation (32) retains all the symmetries of the original integrals in (25)-(27), then the regularization should preserve all symmetries of the theory, as far as three-point and lower functions are concerned. The symmetries of the original

When either $\ell_2$ or $\ell_3$ is nonnegative, the representation has "zeroes" of $\mathcal{O}(\varepsilon_0-\varepsilon_1)$. The limit when all the $\sigma_i$'s vanish exactly corresponds to dimensional regularization. In that case $\varepsilon_0 = \varepsilon_1$ and the aforementioned zeroes become exact, thus providing a rigorous derivation of the vanishing of tadpole integrals in that method.

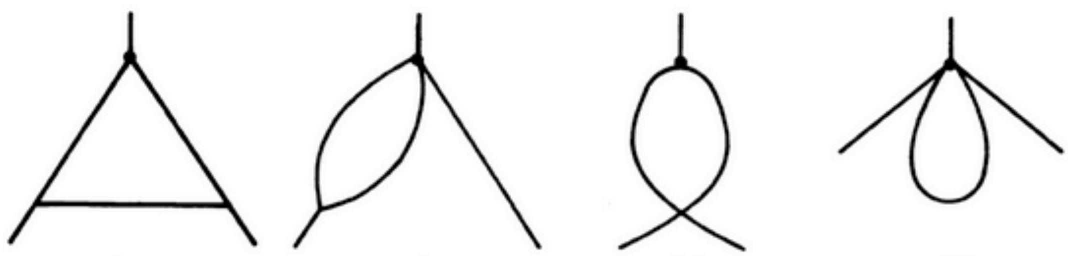
In addition to simple poles, the representation also admits double poles of $\mathcal{O}((1/\varepsilon_0)^2)$ which originate from terms such as $(1/\varepsilon_0)\ \ell n q^2$ in the limit $q^2 \to 0$. As expected, in the calculation of anomalies all simple and double $\varepsilon_0$-poles cancel among themselves.

The choice of linear paths (recall $\varepsilon$, $\sigma_1$ and $\sigma_2$ are by definition of the same order) on the $\varepsilon\sigma$-plane for the limit in (35) is constrained by the possible occurence of $\mathcal{O}(1/\varepsilon_0)$ and $\mathcal{O}(1/\varepsilon_1)$ poles such that paths defined by $\varepsilon_0 = 0$ and $\varepsilon_1 = 0$ are not allowed; any other path is permissible.

Note that, because of the canonicalization of invariant integrals(see paragraph leading to (21)), the exponents $\ell_i$ in (35) are no more directly related to propagators, and even less to that of any species of particle. This makes clear the difference between our method and the analytic method of Speer, and gives reason to why a necessary (but not sufficient) condition for Speer's method [8] to work is to dissociate the exponents from particle species.

## 6. ANOMALOUS SUPERSYMMETRY IDENTITY

We now compute one-loop contributions to the three amplitudes (10), (11) and (12). The four types of graphs that occur in the calculation are given in Figure 1; (a) in the calculation of $J^{(1)}_{5\mu\alpha}$, (a) and (b) in $S^{(1)}_{\mu\alpha\beta}$ and



(a) (b) (c) (d)

all four in $T^{(1)}_{\mu\nu\alpha\beta}$. These graphs are evaluated following the procedure described in section 3 so that each of the (amputated) amplitudes is reduced to a sum of known operators with coefficients being sums of invariant integrals. The tensor reduction is straightforward but can be tedious; here it is carried out with the aid of the algebraic computer program SCHOONSCHIP [40]. The results, with the on-shell conditions (23)

$$\not{\mathcal{D}}\chi = \mathcal{D}_\mu F^{\mu\nu} = 0 \tag{38}$$

incorporated, are [39]

$$p^\mu J^{(1)}_{5\mu\alpha\beta} = -16 i I_u\ \varepsilon_{\alpha\beta\rho\sigma} p^\rho q^\sigma \tag{39}$$

regularization method is used, with the consequential "route-momentum ambiguity" further adding to the myth of anomalies in perturbation theory.

(iii) The tadpole graph (d) in Fig. 1 does not have a vanishing value, but contributes terms proportional to (0,0,-1) to both (44) and (45). Among all integrals this integral is actually the most important one, being directly proportional to the anomalies.

The non-vanishing divergents (39)-(42), (44) and (45) represent anomalies only if they cannot be removed by local counterterms. It can be shown that [39] gauge invariance (all contraction with $q^{\alpha}$ must vanish) demands the constraint

$$I_x = 0 \tag{50}$$

with which the counterterms become uniquely determined. They are

$$(\Delta J_5)_{\mu\alpha\beta} = 4i\ I_u \varepsilon_{\mu\alpha\beta\sigma}(q-r)^{\sigma} \tag{51}$$

$$(\Delta S)_{\mu\alpha} = -6i\ I_u \gamma_5 \gamma^{\sigma} \varepsilon_{\mu\alpha\sigma\beta} q^{\beta} \tag{52}$$

$$(\Delta T)^{\mu}_{\mu\alpha\beta} = -12\ I_u (q\cdot r g_{\alpha\beta} - r_{\alpha} q_{\beta}) \tag{53}$$

In terms of fields, the renormalized currents ($\mathcal{S}^{ren} = \mathcal{S}^{(1)} + \Delta\mathcal{S}$, etc.) satisfy

$$\partial \mathcal{J}_5^{ren} = -2\ I_u\ {}^{*}F_{\mu\nu} F^{\mu\nu} \tag{54}$$

$$\mathcal{S}^{ren}\cdot\gamma = 6\ I_u\ \bar{\chi}\ \sigma^{\mu\nu} \gamma_5\, {}^{*}F_{\mu\nu} \tag{55}$$

$$\mathrm{Tr}(\mathcal{T}^{ren}) = 3\ I_u\ F_{\mu\nu} F^{\mu\nu} \tag{56}$$

where ${}^{*}F_{\mu\nu} = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the dual of $F_{\mu\nu}$. A comparison of (52) and (53) with (43) and (46) reveals that the two anomalies $\mathcal{S}^{ren}\cdot\gamma$ and $\mathrm{Tr}(\mathcal{T}^{ren})$ arise entirely from counterterms. The supersymmetry transformation of the right-hand side of (55) gives

$$(\delta\bar{\chi})\sigma^{\mu\nu} F_{\mu\nu} = i\bar{\varepsilon} F_{\mu\nu} F^{\mu\nu} - \bar{\varepsilon}\gamma_5\ {}^{*}F_{\mu\nu} F^{\mu\nu} \tag{57}$$

$$\bar{\chi}\ \sigma^{\mu\nu}(\delta F_{\mu\nu}) = 2i\bar{\chi}\sigma^{\mu\nu}(\bar{\varepsilon}\mathcal{D}_{\mu}\gamma_{\nu}\chi) = 0 \tag{58}$$

to within a total derivative (because $\bar{\chi}\not{\mathcal{D}} = \not{\mathcal{D}}\chi = 0$). Thus, from (54)-(58)

$$\delta(\mathcal{S}^{ren}\gamma) - 2i\bar{\varepsilon}\ \mathrm{Tr}(\mathcal{T}^{ren}) - 3\bar{\varepsilon}\gamma_5(\partial\mathcal{J}_5^{ren}) = I_u[0] \equiv 0 \tag{59}$$

That is, the identity (1) is satisfied algebraically as a result of supersymmetry, independently of the value of $I_u$.

## 7. LIMIT FOR ANALYTIC REPRESENTATION

The magnitude of the anomalies (54)-(56) is determined by $I_u$, subject to the constraint (50). From (35), (47) and (48), we have

$$I_u = \frac{1}{32\pi^2}\left(\frac{\varepsilon_0-\varepsilon_1}{\varepsilon_1}\right) \tag{60}$$

$$I_x = \frac{1}{32\pi^2}\ \frac{1}{\varepsilon_1}\ (\varepsilon_0 + 3\varepsilon_1 - 2\varepsilon_2) \tag{61}$$

which are undetermined until the limiting path in (35) is chosen. The ellusive nature of these integrals, a reflection of the fact that anomalies depend on finite differences of divergent integrals, is now transparently displayed.

If we take the limit corresponding to dimensional regularization, namely letting $\sigma_1 = \sigma_2 = 0$ so that $\varepsilon_0 = \varepsilon_1 = \varepsilon_2/2 = \varepsilon$. Then $I_x = I_u = 0$. That is, the constraint (50) is satisfied but the anomalies are also incorrectly given vanishing values. (Because $(-1,0,0) \sim (\varepsilon_0-\varepsilon_1)^2/\varepsilon_1 \sim \mathcal{O}(\varepsilon_i)$, $I_u$ depends only on $(0,0,-1)$, the integral intimately related to the tadpole graph Fig. 1(d). The indistinguishability of ultraviolet and infrared singularities ($\varepsilon_0 = \varepsilon_1$), the vanishing of tadpoles and the vanishing of anomalies are closely related shortcomings of dimensional regularization.)

If the integrals are indeterminate, we can ask whether a path in the $\varepsilon\sigma$-plane exists that satisfies $I_x = 0$ and yields the known value [17]

$$I_u = -\frac{1}{16\pi^2} \tag{62}$$

From (60)-(62), it must satisfy

$$\varepsilon_1 = -\varepsilon_0 = \varepsilon_2 \tag{63}$$

The limiting path, from (37), is therefore

$$\varepsilon \equiv 0 \tag{64}$$

$$2\sigma_1 + 3\sigma_2 = 0 \tag{65}$$

The first condition is remarkable. It states that a supersymmetry preserving analytic regularization can be extended for the computation of anomalies only if the number of dimensions is not changed. Note that, had we restricted ourselves to $\varepsilon=0$ from the outset, then the constraint $I_u=0$ alone would have yielded (65), from which (62) and (63) would have followed. With (64) and (65), the anomalies are (in units of $g^2C_2$).

$$\partial \mathcal{J}_5^{\text{ren}} = \frac{1}{8\pi^2}\ {}^{*}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{S}^{\text{ren}}\cdot\gamma = -\frac{3}{8\pi^2}\ \bar{\chi}\ \sigma^{\mu\nu}F_{\mu\nu}$$

$$\mathrm{Tr}(\mathcal{T}^{\text{ren}}) = -\frac{3}{16\pi^2}\ F_{\mu\nu}F^{\mu\nu}$$

We close with some final remarks. We suspect $\varepsilon \equiv 0$ is a general constraint for an analytic regularization if it is to be used to compute anomalies unambiguously. But this can only be confirmed by further studies. Our method differs from that of Speer's because there is not a fixed relation between the exponents $\ell_i$ in (35) and any set of propagators in the theory. We do not believe an analytic regularization (even with $\varepsilon \equiv 0$) without adhering to the tensor reduction procedure described in section 3

38. H.C. Lee and M.S. Milgram, Phys. Lett. 133B(1983)320; Ann. Phys. (NY) 157(1984)408; H.C. Lee, Chi. J. Phys. 23(1985)90.
39. H.C. Lee, Q. Ho-Kim and F.Q. Liu, CRNL preprint, TP-86-VI-11.
40. M. Veltman, "SCHOONSCHIP" (Univ. Michigan, 1984).