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L'ÉNERGIE ATOMIQUE
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A NEW METHOD FOR THE REGULARIZATION OF A CLASS OF DIVERGENT FEYNMAN INTEGRALS IN COVARIANT AND AXIAL GAUGES

Nouvelle méthode de régularisation d'une catégorie d'intégrales Feynman dans les jauges axiales et covariantes

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A NEW METHOD FOR THE REGULARIZATION OF A CLASS OF DIVERGENT FEYNMAN INTEGRALS IN COVARIANT AND AXIAL GAUGES

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Nouvelle méthode de régularisation d'une catégorie d'intégrales Feynman dans les jauges axiales et covariantes

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Résumé

On utilise un ensemble hybride de régularisation dimensionnelle et analytique pour régulariser et découvrir une représentation de fonction-G de Meijer pour une catégorie d'intégrales Feynman divergentes et sans masse se trouvant dans une jauge axiale. Les intégrales se trouvant dans la jauge covariante appartiennent à une sous-catégorie et celles se trouvant dans la jauge à cône léger sont obtenues par continuation analytique. La méthode découple les singularités physiques de l'ultraviolet et de l'infrarouge de la singularité parasite de la jauge axiale mais elle les régularise toutes les trois simultanément. En ce qui concerne la singularité de la jauge axiale, la nouvelle méthode analytique est plus puissante et plus élégante que l'ancienne prescription relative à la valeur principale, mais les deux méthodes possèdent des parties identiques infinies aussi bien que régulières. On démontre que la régularisation dimensionnelle et celle analytique peuvent être équivalentes, en supposant que la première méthode n'a pas d'anomalies-Y5 simulées et que la deuxième conserve une invariance de jauge. La méthode hybride permet d'évaluer les intégrales ayant des puissances arbitraires à nombres entiers pour les logarithmes dans la fonction à intégrer par différentiation à l'égard des exposants. Ces "dérivées exponentielles" engendrent le même ensemble de "polylogarithmes" que ceux engendrés dans les intégrales à boucles multiples selon les théories de perturbation et elles peuvent être utiles pour résoudre les équations selon les théories de non-perturbation. La relation étroite qui existe entre la méthode des dérivées exponentielles et la prescription de 'tHooft et Veltman pour traiter les différences de chevauchement est notée. On démontre que les deux méthodes engendrent des fonctions n'ayant pas de parties logarithmiques infinies non renormalisables. C'est pourquoi les théories de non- perturbation exprimées en termes de dérivées exponentielles sont renormalisables. D'intrigants rapports existant entre les théories de non-perturbation et les exposants non intégraux sont notés.

Des extraits de ce rapport ont été publiés dans la littérature (références 50-56). Le but de ce rapport est de présenter un texte unifié et de commenter certains aspects de la théorie qui ne sont pas traités dans la littérature.

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Abstract

A hybrid of dimensional and analytic regularization is used to regulate and uncover a Meijer's G-function representation for a class of massless, divergent Feynman integrals in an axial gauge. Integrals in the covariant gauge belong to a subclass and those in the light-cone gauge are reached by analytic continuation. The method decouples the physical ultraviolet and infrared singularities from the spurious axial gauge singularity but regulates all three simultaneously. For the axial gauge singularity, the new analytic method is more powerful and elegant than the old principal value prescription, but the two methods yield identical infinite as well as regular parts. It is shown that dimensional and analytic regularization can be made equivalent, implying that the former method is free from spurious γ_5-anomalies and the latter preserves gauge invariance. The hybrid method permits the evaluation of integrals containing arbitrary integer powers of logarithms in the integrand by differentiation with respect to exponents. Such "exponent derivatives" generate the same set of "polylogs" as that generated in multi-loop integrals in perturbation theories and may be useful for solving equations in nonperturbation theories. The close relation between the method of exponent derivatives and the prescription of 'tHooft and Veltman for treating overlapping divergencies is pointed out. It is demonstrated that both methods generate functions that are free from unrenormalizable logarithmic infinite parts. Nonperturbation theories expressed in terms of exponent derivatives are thus renormalizable. Some intriguing connections between nonperturbation theories and nonintegral exponents are pointed out.

Excerpts from this report can be found in the published literature (refs. 50-56). The purpose of this report is to provide a unified exposition, and discuss some aspects of the theory that do not appear in the literature.

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I. INTRODUCTION

Dimensional regularization[1-4] is a powerful tool for regulating the ultraviolet[1] and infrared[2] divergent integrals intrinsic to quantum field theories. Because the method preserves gauge invariance and at the same time provides the easiest way to isolate the infinite part as well as the leading logarithmic term of divergent Feynman integrals, from its conception it has been extensively used in the study of renormalization[1-5] and the dominant asymptotic behaviour of gauge theories in perturbation calculations[6].

The analytical properties of a dimensionally regulated integral do not appear to have been fully explored however, particularly for integrals in an axial gauge,[7] which are especially difficult to evaluate. The chief advantage in choosing an axial gauge is that the Faddeev-Popov ghosts[8] that are otherwise required to uphold Ward identities[9] in non-Abelian theories are decoupled from the physical fields. This greatly simplifies calculations and makes practicable otherwise intractable calculations in theories such as quantum gravity. Other advantages of the axial gauge are that it yields mass factorization[10], and in hard quantum chromodynamic processes, a judicious choice of the special planar gauge[11] causes virtual gluons to be effectively physical, i.e., tranversely polarized.

The price one pays for the convenience of the axial gauge is that integrals involving the propagator now have, in addition to the ultraviolet and infrared divergencies that they may otherwise possess, unphysical or "spurious" singularities that are associated only with the axial gauge. In the literature, integrals that suffer from such axial gauge singularities have been generally (but not exclusively) regularized with the principal value prescription[12]. Recently it has been shown[13,49] that this prescription, used in conjunction with the method of dimensional regularization, yields well defined and consistent results for the infinite parts of axial gauge integrals of two-point functions in Yang-Mills theories and quantum gravity at the one-loop level. The prescription, however, is sufficiently cumbersome that the evaluation of any axial gauge integral is a substantial undertaking. Moreover, the evaluation of the finite (or regular) parts of these integrals, other than the leading logarithmic term, is difficult with this prescription.

In this paper we propose a method, based on dimensional regularization (the dimension of space-time is generalized to a continuous variable) and analytic regularization (exponents are generalized to continuous variables)[14], for calculating a very general class of massless divergent integrals in the axial gauge; integrals in the covariant gauge, which are free from spurious singularities constitute a subset of the class. Specifically, in our method spurious singularities are dealt with by analytic regularization, not by the principal value prescription. It will be demonstrated that the proposed new method is more powerful and elegant than the old one. At the same time, by means of constructing an axial gauge "regulator" for the the principal value prescription it will be shown that both methods yield identical results, for the finite as well as the infinite parts.

On the broader perspective of regularization in general, not restricted to that of axial gauge singularities, we observe that analytic regularization and dimensional regularization, for singularities that can be regularized by the two methods separately, yield identical results, apart from certain terms that can be identified and subtracted. One type of singularity that cannot be regularized by dimensional regularization alone but can be regularized by the other method is the axial gauge singularity. We have not encountered any type of singularity that can be regulated by dimensional regularization but not by analytic regularization. In this sense, at least for the evaluation of the class of integrals considered, dimensional regularization is in fact redundant. It must be emphasized that we do not advocate the replacement of dimensional regularization by analytic regularization. The reason is obvious, for in situations where the former method works, it is much the superior one requiring the generalization of only one integer - the number of dimensions - into a continuous variable. The latter method requires the generalization of several integer exponents. On the other hand, because dimensional regularization has some known limitations[1,3,4,15] arising from the ambiguity of doing algebra in continuous dimensions - the most famous one being that related to the Bell-Jackiw-Adler anomaly[16] - the recognition that analytic and dimensional regularization are equivalent is important; since analytic regularization does not affect the algebra, it is clear that (for situations where dimensional regularization will work) one should do all the algebra in 4-dimensions to reduce the integrand to a function of scalars in Euclidean (or Minkowski) space before regulating the integral by dimensional regularization. This is precisely the strategy adopted in the recently proposed method of dimensional reduction[17]. This being the case, we further demonstrate that it is unnecessary to restrict this method to spaces of less than four dimensions[17]. The fact that we regulate only Feynman integrals also allows us to assert that our analytic regularization preserves gauge invariance. This is in contrast to Speer's[14)] analytic method of regulating propagators which has the appearance of not preserving gauge invariance[3)].

The class of integrals we shall study is defined by

$$S_{2\omega}(p,n;\kappa,\mu,\nu,s) \equiv \int d^{2\omega}q[(p-q)^2]^{\kappa}(q^2)^{\mu}(q\cdot n)^{2\nu+s}, \qquad (1.1)$$

where ω, κ, μ, ν are arbitrary, continuous variables, $s = 0$ or 1, p is an external momentum, and n is an external vector used to define the axial gauge condition $A\cdot n = 0$; A is the gauge field. For simplicity we choose to work in a (2ω-dimensional) Euclidean space; the conversion of our result to Minkowski space follows the usual procedure[18)]. Whenever the situation allows, we shall suppress the subscript and/or some of the variables of the function on the left-hand-side of (1.1). Thus we may write $S_{2\omega}(p,n)$, $S(p,n)$, or simply S, which we shall call an S-integral. The class of integrals (1.1) is the generalization of the class of "primal" four-dimensional integrals $S_4(p,n;K,M,N,s)$ with integer exponents. Our main result is the discovery of a closed-form expression for the S-integrals that is a well-defined and analytic function of ω, κ, μ, ν and the scalar products p^2, $p\cdot n$ and n^2.

When $\omega=2$ and κ, μ and ν are integers, the S-integrals reduce to primal integrals in perturbation calculations at the one-loop level for two-point functions in massless Yang-Mills theories and quantum gravity.[13)] The subset with $\nu = s = 0$ are the corresponding integrals in covariant gauges.

Our motivation for letting the exponents κ, μ and ν be continuous is:

(a) It is necessitated by the method of analytic regularization.

(b) The method of dimensional regularization generates continuous exponents in multi-loop integrals in perturbation theories.[1]

(c) Having κ, μ and ν continuous allows us to generate, and regulate integrals with integrands containing powers of $\ell n(p-q)^2$, $\ell n q^2$ and $\ell n(q \cdot n)$, by means of taking partial derivatives of the S-integral with respect to κ, μ and ν respectively. Such integrals arise in multi-loop calculations.

(d) Integrals with noninteger exponents may appear in nonperturbation calculations even when they do not appear in perturbation calculations.

If our sole purpose were to regulate the axial gauge singularity (by analytic regularization) it would only be necessary to generalize the exponent ν; singularities associated with the exponents κ and μ can be more expediently regulated by dimensional regularization. However, by generalizing all κ, μ and ν we are able to establish the relation between analytic regularization, dimensional regularization and dimensional reduction discussed earlier.

To be more specific about the relevance of the S-integrals in nonperturbation calculations consider the example of the Schwinger-Dyson equations[19] for the reduced gluon propagator, Z, which respects Ward identities:

$$\begin{aligned} Z^{-1} &= 1 + g^2 \int d^4q \; K(q,p,n) \; Z(q,n) \\ &\quad + \text{integral with integrand quadratic in } Z, \end{aligned} \tag{1.2}$$

where g is the coupling constant and K is a known kernel.[20] The integral posseses ultraviolet, infrared and axial gauge singularities. However, since Z is unknown until after (1.2) is solved, the normal procedure of renormalization by dimensional regularization is impractical. Alternative renormalization (such as subtraction) schemes[5] either do not preserve gauge invariance or present major numerical problems,[20] or both. Now if we write $Z(p,n)$ as a product of continuous powers of p^2 and/or $p\cdot n$ and polynomials in p^2 and $p\cdot n$, or a sum of such, then all integrals in (1.2) are reduced to S-integrals allowing the renormalization program to be carried out as usual[21] (i.e. as in perturbation calculations). Eq. (1.2) may then reduce, by truncating to the appropriate order, to a finite set of algebraic equations. This is similar to the expansion method for solving an integral equation. In this way a knowledge of the S-integrals opens the way to carry out the renormalization program in nonperturbation calculations. We note that it is precisely because of the lack of a viable renormalization method that most nonperturbation calculations have circumvented the need for regularization by either quantizing around classical solutions[22] and restricting calculations at the tree level, or by using a method where divergencies are prevented from occurring, such as the discretization of space-time in lattice gauge theories.[23,24]

When derivatives with respect to κ, μ and ν are taken on both sides of (1.1), one obtains

$$S^{j_1,j_2,j_3} \equiv \left(\frac{d}{d\kappa}\right)^{j_1}\left(\frac{d}{d\mu}\right)^{j_2}\left(\frac{d}{d\nu}\right)^{j_3} S = \int d^{2\omega}q \ \cdots \ \ln^{j_1}(p-q)^2 \ln^{j_2}(q^2) \ln^{j_3}(q\cdot n)^2 . \tag{1.3}$$

We designate such derivatives as "exponent derivatives", and refer to the right-hand side as S-integrals with logarithmic factors. In perturbation calculations of quantum field theories, the occurrence of logarithms is associated with divergent integrals; N-loop integrals may yield logarithms up to the N^{th} power. Thus, linear combinations of $S^{j_1,j_2,j_3}(\kappa,\mu,\nu)$ where κ, μ and ν are integers and $j_1+j_2+j_3 \leq N-1$, represent Feynman integrals at the N-loop level. This recognition provides the following insight into nonperturbation calculations, again taking the Schwinger-Dyson eq. (1.2) as an example: If $Z(q,n)$ in the integrand is represented by a polynomial in q^2 and $q \cdot n$, then effects up to the one-loop level are included. If $Z(q,n)$ contains terms with factors of up to $(N-1)$ powers of logarithms in q^2 and $q \cdot n$, then effects up to the N-loop level are included. We therefore give this meaning to the truncation "order" of nonperturbation calculations: in an N^{th} order calculation, the integrand has factors of $(N-1)$ powers of logarithms. This implies that an N^{th} order nonperturbation calculation can be reduced to one-fold (2ω-dimensional) integrals with logarithmic powers of order $(N-1)$, as opposed to an N^{th} order perturbation calculation involving N-fold integrals without logarithms.

Another issue regarding Feynman integrals with logarithmic integrands concerns the problem of overlapping divergencies[25]: how does one show that in N-loop divergent integrals an infinite term with a logarithmic dependence on the external momentum will never emerge? Such an infinite term cannot be renormalized and therefore must not appear in a renormalizable theory.[5] 'tHooft and Veltman[1] demonstrated that in perturbation theory such infinite terms are cancelled by the subtraction of counterterms. We shall prove that, based on the analytical properties of the S-integrals and its exponent derivatives, calculations in nonperturbation theories can be made free from such infinite terms.

The rest of the paper is organized as follows. In section 2 we present our main result, relating the S-integrals to a Meijer's G-function[26)] which is a transcendent of hypergeometric functions and is a well-defined, analytical function of ω, κ, μ, ν and $y \equiv (p\cdot n)^2/p^2n^2$. The derivation, details of which are given in two Appendices is naturally divided into two steps: the first regulating the S-integral to "canonical" form (Appendix A), and the second identifying the canonical integral as a Meijer's G-function (Appendix B). The divergent nature of the primal S-integral is revealed in the contour integral representation of the G-function by pinches of the contour at certain values of the variables. It will be shown that the infinite part is a certain power of p^2 times a terminating polynomial in y, and the finite or regular part is the sum of an explicit series in y if $|y| \leq 1$, or a different series in 1/y if $|y| > 1$, plus logarithmic terms. In the case of covariant gauges, i.e. when $\nu=s=0$, all series collapse to a form independent of y, as expected, since this integral must be independent of n. The G-function representation treats the cases of space-like ($n^2<0$) and time-like ($n^2>0$) gauges equally well. The light-cone gauge ($n^2=0^+$), corresponding to the limit $y\to+\infty$, is a special case of the continuation to $|y| > 1$.

In section 3 and Appendix C we show that our analytic regularization of axial gauge singularities yields a result which is identical to that given by the principal value prescription[12)]. We show, by explicit construction, that the principal value prescription for an integral with axial gauge singularities of arbitrary order yields a result that can be compactly expressed as a polynomial in a differential operator operating on a sum of G-functions.

In section 4, we use the analytical properties of the G-function to prove several theorems describing the structure of the pole and logarithmic terms of the S-integral and its exponent derivatives. It is shown that as far as this structure is concerned, the set of functions spanned by the zeroth, first, •••, $(N-1)^{th}$ order exponent derivatives of one-loop integrals is equivalent to the set spanned by the one, two, •••, N-loop integrals in perturbation expansions; it is the set of polynomials in p^2 times a polynomial containing up to N powers of $\ln p^2$ – "polylogs"[27] of order N.

Different regularizations of divergent integrals yield identical infinite parts but generally differing finite parts. We identify subtractions that render the methods of dimensional and analytic regularization completely equivalent (thereby eliminating the need for extending the Dirac algebra to continuous dimensional spaces), and those that render exponent derivatives of S-integrals free from logarithmic infinite parts. The latter implies that nonperturbative theories based on such exponent derivatives are renormalizable. For completeness, in Appendix D we demonstrate that the 'tHooft-Veltman prescription[25] eliminates all logarithmic infinite parts in all multi-loop integrals in perturbation theories.

To illustrate the power of the G-function representation, in Section 5, we present several analytic examples. The infinite parts of these cover all those we have encountered in the literature. We give several examples for the exponent derivatives of the S-integral. The

G-function representation also provides new insights concerning covariant gauge integrals, tadpole integrals and the very special properties of the light-cone gauge. We classify all primal S-integrals and present their infinite and finite parts compactly in a Table.

In Section 6 we comment on the intriguing implications when physical amplitudes in nonperturbation theories are represented by S-integrals with non-integral exponents.

Section 7 is a summary.

2. ANALYTIC REPRESENTATION OF THE S-INTEGRAL

2.1 General Considerations

The integral under scrutiny may be formally treated as a function of several complex variables. To justify this approach, consider the integral 1.1 defined by

$$S_{2\omega}(p,n;\kappa,\mu,\nu,s) = \int d^{2\omega}q(q^2)^{\mu}(q\cdot n)^{s}[(q\cdot n)^2]^{\nu}[(p-q)^2]^{\kappa} \tag{2.1}$$

where the integration extends over a Euclidean space generalized to 2ω dimensions[1,3,4) in a manner discussed in Appendix A. To guarantee that this integral has meaning, it suffices to choose (continuous) ω compatible with arbitrary (continuous) variables (μ, ν and κ with $s = 0$ or 1) such that the integral representation (2.1) exists. In comparison and contrast to 'tHooft and Veltman[1)] who regularize only ultraviolet divergencies, it is not sufficient to demand that $\mathrm{Re}(\omega)$ be arbitrarily large and negative; in the definition (2.1) there exist infra-red and axial gauge (spurious) singularities with which to contend. However, a region in (ω,μ,ν, κ)-space exists such that (2.1) is well defined. So, it is enough[28)] to devise a representation for the integral (2.1) valid for a larger range of the variables but with some overlap with the region of existence.

In Appendices A and B, the following result is derived:

$$S_{2\omega}(p,n;\kappa,\mu,\nu,s) = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa+\nu}(n^2)^{\nu}\,\Gamma(s+\nu+1/2)(p\cdot n)^{s}}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)\,\Gamma(2\omega+2\nu+\mu+\kappa+s)}$$

$$\times\; G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1-\omega-\mu-\nu-s,\ \omega+\mu+\nu+\kappa+1,\ \nu+1;\\ 0,\ \omega+\nu+\kappa;\ 1/2-s\end{matrix}\right), \tag{2.2}$$

where $y = (p\cdot n)^2/p^2n^2$, and G is Meijer's G-function,[26] a compact notation for a function which can be represented either as a contour integral (2.8) or as a sum of two generalized hypergeometric functions as in (2.7). In the derivation of (2.2) a number of conditions are required ((B.6) and (A.8)), which collectively delineate the region in which the integral (2.1) exists. The conditions are

$$-1/2 - s < \mathrm{Re}(\nu) < 0, \tag{2.3a}$$

$$\mathrm{Re}(\mu) < 0, \tag{2.3b}$$

$$\mathrm{Re}(\kappa) < 0, \tag{2.3c}$$

$$|y| \leq 1, \tag{2.3d}$$

$$-\mathrm{Re}(\nu) + \mathrm{Max}\left(\mathrm{Re}(-s-\mu, -\frac{\mu+\kappa+s}{2})\right) < \mathrm{Re}(\omega) < -\mathrm{Re}(\mu+\nu+\kappa). \tag{2.3e}$$

Of course, the right-hand-side of (2.2) is well-defined for all values of the variables and the conditions (2.3) may be dispensed with.

Since κ, μ, ν and ω are thought of as being independent (real) variables, it is convenient to introduce some simplifying notation:

$$\kappa = K + \rho, \tag{2.4a}$$

$$\mu = M + \sigma, \tag{2.4b}$$

$$\nu = N + \tau, \tag{2.4c}$$

$$\omega = 2 + \varepsilon, \tag{2.4d}$$

where K, M and N are integers and ρ, σ, τ and ε are variables which will eventually be made arbitrarily small. Furthermore, we define the indices

$$\alpha_0 = - \mu - \nu - s - \omega, \tag{2.5a}$$

$$\alpha_1 = \kappa + \mu + \nu + \omega, \tag{2.5b}$$

$$\alpha_2 = \nu, \tag{2.5c}$$

$$\beta_1 \equiv \alpha_b = \kappa + \nu + \omega, \tag{2.5d}$$

composed of integral parts:

$$A_0 = - M - N - s - 2, \tag{2.5e}$$

$$A_1 = K + M + N + 2, \tag{2.5f}$$

$$A_2 = N, \tag{2.5g}$$

$$B_1 = K + N + 2, \tag{2.5h}$$

and epsilons:

$$\varepsilon_0 = -\alpha_0 + A_0 = \sigma + \tau + \varepsilon, \tag{2.5i}$$

$$\varepsilon_1 = -\alpha_1 + A_1 = -\rho - \sigma - \tau - \varepsilon, \tag{2.5j}$$

$$\varepsilon_2 = -\alpha_2 + A_2 = -\tau, \tag{2.5k}$$

$$\varepsilon_b = \beta_1 - B_1 = \rho + \tau + \varepsilon, \tag{2.5ℓ}$$

in terms of which

$$S = \frac{\pi^{\omega}(p^2)^{\alpha_1}(n^2)^{\alpha_2}(p\cdot n)^s\Gamma(\alpha_2+s+1/2)}{\Gamma(\beta_1-\alpha_0)\Gamma(\beta_1-\alpha_1)\Gamma(-\alpha_0-\alpha_1-s)\Gamma(-\alpha_2)}$$

$$\times\, G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}1+\alpha_0, 1+\alpha_1, 1+\alpha_2; \\ 0,\ \beta_1;\ 1/2-s\end{matrix}\right). \tag{2.6}$$

The G-function is symmetric under any permutation among α_0, α_1, and α_2. S has less symmetry because of the factors in (2.6) exterior to the G-function; aside from the factor $(p^2)^{\alpha_1}$, S is symmetric under the interchange $\alpha_0 \leftrightarrow \alpha_1$. The factor $(p^2)^{\alpha_1}$ reflects the overall dimension of S save the unimportant factor $(p \cdot n)^s$. From (2.1) and (2.5a-c), the indices α_0, α_1 and α_2 can be recognized as relating to the infrared, ultraviolet and axial gauge singularities respectively of the original S-integral, and shall be referred to as such. Significantly, with one exception, ω appears in (2.6) only via the indices of (2.5), i.e., in linear combinations with κ, μ and ν, and always with a relative coefficient ± 1. The exception is the factor π^ω, which has no bearing on the singular properties of S; unless otherwise mentioned, we shall ignore this factor in our discussion.

2.2 Regularization

Consider the primal integral $S_4(p,n;K,M,N,s,)$, whose integral representation (2.1) may or may not exist. The S integral (2.1) with arbitrary parameters is a generalization of the primal integral, and may be analytically continued to all values of the parameters using (2.2). We define the regularized primal integral to be the right-hand-side of (2.2) in the limit ε, ρ, σ, $\tau \to 0$.

The regularization process is intimately connected with the manner in which ρ, σ, τ and ε are set to zero. In the first place we wish to regularize the axial gauge singularity ($q \cdot n = 0$) which lies on the path of integration. To achieve this, we use analytic regularization (A.1) by letting ν become a continuous (complex) variable, and requiring that ν lie in the range (2.3a) in which (2.1) is defined; the axial gauge singularity has become integrable. In the final result (2.2) we consider values of ν outside the original range of definition, a process justified by the principles of analytic continuation which allows us to uniquely continue a function defined over a region, but not over a set of isolated points (integers).[28] It is significant that this procedure is independent of ω, reflecting the fact that the axial gauge singularity is spurious. The result (2.2) is a meromorphic function of ν although the original integral is singular if $A_2 \leq -(s+1)/2$. In (2.2) the G-function is singular whenever ν is a non-negative integer, but this singularity is cancelled by the zero of $1/\Gamma(-\nu)$. So, S has no singularities when ν is an integer and the limit $\tau \to o$ can be evaluated before all others - the spurious singularity has been regulated away. However, because ν is a continuous variable it is permissible to take derivatives with respect to ν in order to evaluate exponent derivatives - integrals with integrands containing powers of $\ln(q\cdot n)^2$.

The regularization of the infrared and ultraviolet divergencies is somewhat more complicated, since these are end-point singularities and are therefore closely connected with the dimensionality of the integral. We regularize these divergencies respectively by initially choosing

$$\alpha_0 \geq - s/2,$$
$$\alpha_1 \geq - s/2,$$

and analytically continuing the result (2.2) in either ω (dimensional regularization) or μ and κ (analytic regularization) or both (hybrid).

In the method of dimensional regularization[1-4] one is limited to regulating the axial gauge singularity with the principle value prescription[12] (cf. Sec. 3). Insofar as the other singularities are concerned, one sets $\rho = \sigma = 0$ at the outset, performs analytic continuation in ω by letting $\varepsilon \to 0$ and identifies the terms of $O(1/\varepsilon)$ as the infinite parts of S. This method does not permit the computation of derivatives with respect to κ, μ and ν, nor the evaluation of integrals with M and K outside the limits given in (2.3b,c) - M and K must be negative integers in dimensional regularization.

In the method of analytic regularization,[14] which must be invoked if exponent derivatives are desired, $\varepsilon=0$ at the outset and the infinite parts of S appear as $O(1/\sigma)$ and/or $O(1/\rho)$ terms. As described earlier, the would-be axial gauge singularities of $O(1/\tau)$ do not appear.

In practice we choose the hybrid regularization which posesses the power of analytic regularization - it allows the simultaneous regularization of infrared, ultraviolet and axial gauge singularities - but retains the simplicity of dimensional regularization: allow all ε, ρ, σ and τ to be non-zero until after the S integral and/or exponent derivatives have been evaluated, then set $\rho = \sigma = \tau = 0$ and evaluate the limit $\varepsilon \to 0$.

A fundamental observation can now be made by inspecting the G-function in (2.6): all singularities of S due to divergencies of the original integral arise from singularities of the G-function - poles in the complex $(\omega, \kappa, \mu, \nu)$ space - that occur whenever the difference between one of the top three parameters and one of the first two bottom parameters is a positive integer. The fact that S depends on ε, ρ and σ through the indices of (2.5) ensures that coefficients of the $O(1/\varepsilon)$ terms and those of the

$O(1/\rho)$ and $O(1/\sigma)$ terms in S are the same, although those of higher order ($O(1)$, $O(\varepsilon)$, $O(\rho)$, $O(\sigma)$, etc.) terms may differ. This is expected because there is no unique generalization of a function defined over a set of integers. For example, any regular function proportional to ρ, σ, τ, or ε can be arbitrarily added to S with no effect on S, but with a profound effect on its exponent derivatives.

Finally, we consider the G-function in (2.2) as a function of y. The analytic continuation of a G-function outside the circle $|y| \leq 1$ is well-defined, and in the case considered the result is another G-function valid for $|1/y| < 1$. In particular the point $1/y = 0^+$ corresponding to the case $n^2 = 0^+$ is accessible. This special case leads to representations useful in the light-cone gauge, to be discussed in Sections 2.2, 5.1 and 5.4.

The nature of the singularity on the circle $|y| = 1$ may be investigated by writing[26] the G-function as a sum of two ${}_3F_2$'s, and evaluating the difference between the second and first set of parameters.[29] Explicitly,

$$S = \frac{\pi^{\omega}(p^2)^{\mu+\nu+\kappa+\omega}\ (n^2)^{\nu}(p\cdot n)^{s}\,\Gamma(\nu+s+1/2)}{\Gamma(-\kappa)}$$

$$\times \Big\{\frac{\Gamma(\nu+\kappa+\omega)\,\Gamma(\omega+\mu+\nu+s)\,\Gamma(-\mu-\nu-\kappa-\omega)}{\Gamma(1/2+s)\,\Gamma(-\mu)\ \ \Gamma(\mu+2\nu+\kappa+s+2\omega)}\ {}_3F_2\left(\left.{\omega+\mu+\nu+s,-\omega-\kappa-\mu-\nu,-\nu \atop 1/2+s,1-\omega-\kappa-\nu}\right| y\right)$$

$$+\frac{\Gamma(-\omega-\nu-\kappa)\,\Gamma(\kappa+\omega)}{\Gamma(1/2+\nu+\kappa+s+\omega)\,\Gamma(-\nu)}\ y^{\nu+\kappa+\omega}\ {}_3F_2\left(\left.{2\nu+\kappa+\mu+s+2\omega,-\mu,\kappa+\omega \atop 1+\nu+\kappa+\omega,1/2+\nu+\kappa+s+\omega}\right| y\right)\Big\} \tag{2.7}$$

which leads to the condition[29] for convergence at $y = 1$:

$\omega < 3/2.$

This condition can never be satisfied in spaces of dimensionality greater than 2, and so the point y=1 appears to be a regular singular point. However, the nature of a G-function in the neighbourhood of the point y=1 is not adequately analyzed in the literature.[26] Physically there is nothing special about the point y=1 except in real Euclidean spaces, where $|y|\leq 1$ usually this means that a branch cut appears at this point. Although each of the hypergeometric functions in (2.7) possesses a cut singularity starting at y=1, we speculate that the general combination of the two does not. This is illustrated by example in Section 5.1. A cut does exist along the negative y axis (y spacelike) starting at y=0 however, as predicted by G-function theory[26]. This is manifested by ℓny behaviour in expansions about y=0.

2.3 Contour Integral Representation for $|y| < 1$

We may write the G-function in its contour integral representation,[30]

$$S = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa+\nu}(n^2)^{\nu}\Gamma(s+\nu+1/2)(p\cdot n)^s}{\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\kappa)\Gamma(2\nu+\mu+\kappa+s+2\omega)}$$

$$\times \frac{1}{2\pi i}\int_L dt\, y^t \frac{\Gamma(-t)\Gamma(\omega+\nu+\kappa-t)\Gamma(\mu+\nu+s+\omega+t)\Gamma(-\mu-\nu-\kappa-\omega+t)\Gamma(-\nu+t)}{\Gamma(1/2+s+t)} \tag{2.8}$$

where the contour L encloses the poles of the first two gamma functions, and excludes the others. The situation is depicted schematically in

Fig. 1. Note that inside the contour one string of poles of the integrand in (2.8) is fixed at the non-negative integers whereas a second (β_1) string of poles approaches all but a finite number of the first only when ν and κ are integers and $\varepsilon\to0$.

Exterior to the contour, we find "fixed" (independent of ε) poles of the integrand pinching the contour as $\tau\to0$ and "moving" (ε-dependent) poles also pinching the contour as ρ, σ, τ and ε approach zero. Such pinches will be reflected as pole singularities of the contour integral at $\varepsilon = \rho = \sigma = \tau = 0$. In addition there exist both fixed and moving zeros from the gamma functions exterior to the contour integral, acting to reduce the overall degree of singularity of S. The result is that S has simple poles at $\varepsilon = \rho = \sigma = 0$ in the $\varepsilon, \rho, \sigma$ plane, verifying our earlier claim that S is free from axial gauge singularities and is regular at $\tau=0$.

Alternatively, S may be viewed as a function of the indices of (2.5), as in (2.6). If we further define

$$\varepsilon_3 \equiv \varepsilon_b + \varepsilon_2, \tag{2.9}$$

then we find S has simple poles at $\varepsilon_i = 0$ in the ε_i-planes, $i = 0,1,3$. From Fig. 1, we observe that pinches in the contour have their genesis in three strings of exterior poles (α_0, α_1 and α_2) extending to the left and two strings of interior poles (one labelled β_1, the other being the non-negative integers) extending to the right. A first kind of pinch singularity of the contour integral occurs whenever $\mathrm{Re}(\alpha_i)\geq0$ $(i = 0,1,2)$ and a second kind occurs whenever an α_i pinches β_1. The singularities generated by the α_2 (axial gauge singularity), $\alpha_0-\beta_1$ and $\alpha_1-\beta_1$ pinches are cancelled by corresponding zeros of the gamma functions exterior to the contour integral (see (2.6)). The three surviving singularities appear as poles of S and reflect the physical divergencies in the original integral (2.1) - infrared (α_0), ultraviolet (α_1) and that generated at the point q = p (α_2-β_1).

By studying the interaction between the pinches and zeros as they coalesce, it is possible to demonstrate that the singularities of S are at most simple poles in the (ω, κ, μ, ν) space (cf. (4.7a)). We caution against interpreting the two strings of interior poles of the integrand as a single string of dipoles, except at points where pinches do not occur, in which case the dipole interpretation simplifies computation.

The "overlap region" where the pinches reside contains a finite number of poles. Thus writing the integral in the form

$$S_{2\omega}(p,n) = \sum_{i=0,1,3} \frac{I_i(y,p,n)}{\varepsilon_i} + R(y,p,n) \tag{2.10}$$

where $I_i(y,p,n)$ are the numerators of the divergent (or infinite) part and $R(y,p,n)$ is the regular (or finite) part we see that $I_i(y,p,n)$ is a function of y with a finite number of terms (possibly a polynomial), since only pinches in the overlap region contribute to it. The regular part R may consist of an infinite series in y restricted to $|y| < 1$ due to poles of the integrand starting at $t > \mathrm{Max}(A_0,A_1,A_2,0)$ and extending to $t\to+\infty$, plus a finite number of higher order derivative terms surviving from the overlap region. I_i and R are also regular functions of the epsilons with leading O(1) terms. The pole structure of S is discussed in more detail in Section 4. Specific examples of S and its general expansion in ε are given in Section 5.

2.4 Contour Integral Representation for $|y| > 1$ and the Light-Cone Gauge

From the theory of G-functions[31)] it is possible to analytically continue the representation (2.2) to the region $|y| > 1$. The result is

$$S = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa}(p\cdot n)^{2\nu+s}\,\Gamma(s+\nu+1/2)}{\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\kappa)\Gamma(2\omega+2\nu+\mu+\kappa+s)}$$

$$\times\, G^{3,2}_{3,3}\left(\frac{1}{y}\middle|\begin{matrix}1+\nu,1-\omega-\kappa;1/2+\nu+s\\ 0,-\omega-\mu-\kappa,\omega+\mu+2\nu+s;\end{matrix}\right) \tag{2.11}$$

which has the contour integral representation

$$S = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa}(p\cdot n)^{2\nu+s}\,\Gamma(s+\nu+1/2)}{\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\kappa)\Gamma(2\nu+\mu+\kappa+s+2\omega)}$$

$$\times\, \frac{1}{2\pi i}\int_L dt\; y^{-t}\,\frac{\Gamma(-t)\Gamma(-\mu-\kappa-\omega-t)\Gamma(\mu+2\nu+s+\omega-t)\Gamma(-\nu+t)\Gamma(\kappa+\omega+t)}{\Gamma(1/2+\nu+s-t)} \tag{2.12}$$

illustrated schematically in Fig. 2. The same comments hold as for (2.8) except that L now encloses three strings of interior poles extending to the right, and excludes two strings of moving and fixed exterior poles extending to the left. The interior (exterior) poles are the former exterior (interior) poles with opposite sign translated by ν. Again there is an overlap region pinching only a finite number of poles, so the contribution to the divergent terms $I_i(y,p,n)$ can at most contain a finite sum of $1/y$ terms.

Since finite sums are their own analytic continuation, it follows that all representations of $I_i(y,p,n)$ valid for $|y| < 1$ will be valid for $|y| > 1$; the same is true for the finite number of survivors from the overlap region that contribute to $R(y,p,n)$; these terms will contain factors of $1/y$ and ℓny. It is thus sufficient to evaluate the analytic continuation of any infinite series in $R(y,p,n)$ to obtain representations for $S_{2\omega}(p,n;\kappa,\nu,\mu,s)$ valid for all values of y. This is done explicitly by example in Sec. 5.1, and in general in Sec. 5.4.

The condition $n^2 = 0$ defines the light-cone gauge[32]. In (2.11) all n^2 dependence resides in y, so to study this gauge it is sufficient to examine the limit $(1/y) \to 0^+$. The lead term in the string of interior poles gives the dominant $(1/y)$ dependence; in order that S may approach a finite limit as $1/y \to 0^+$ we require that both moving interior poles lie to the right of the origin. The simultaneous conditions are

$$-\mu-2\nu-s < \omega < -\mu-\kappa \tag{2.13}$$

in which case the leading $(1/y)$ dependence will be like $(1/y)^0$ because of the pole at the origin. The conditions (2.13) are necessary, but not sufficient since they do not preclude ℓny dependence associated with pinches, (cf. Section 5.1) and a special limiting process is required for this gauge[54,55]. With this special limit, we find

$$S(p,n;\kappa,\mu,\nu,s)\bigg|_{n^2=0} = \frac{\pi^\omega (p^2)^{\omega+\kappa+\mu}(p\cdot n)^{2\nu+s}\,\Gamma(\omega+\kappa)\,\Gamma(\omega+\mu+2\nu+s)\,\Gamma(-\omega-\kappa-\mu)}{\Gamma(-\kappa)\,\Gamma(-\mu)\,\Gamma(2\omega+\kappa+\mu+2\nu+s)} \tag{2.14}$$

In ref. 56 we have demonstrated that analytic regularization and the special limit preserves both gauge invariance and the desirable and simplifying feature of the light-cone gauge - allowing the $O(n^2)$ term in the propagator of the gauge particle to be dropped at the outset.

2.5 The point y = 0

In the special[33] "gauge" $p\cdot n = 0$, we see from (2.6) or (2.7) that the limit $y=0^+$ is easily obtained; the integral may be finite in this limit if we analytically continue from the region

$$\omega+\kappa+\nu > 0 \,. \tag{2.15}$$

With this condition, we find[55]

$$S(p,n;\kappa,\mu,\nu,0)\bigg|_{p\cdot n=0} = \frac{\pi^\omega (p^2)^{\mu+\nu+\kappa+\omega}(n^2)^\nu\,\Gamma(\nu+1/2)\,\Gamma(\nu+\kappa+\omega)\,\Gamma(\omega+\mu+\nu)\,\Gamma(-\omega-\mu-\nu-\kappa)}{\Gamma(1/2)\,\Gamma(-\kappa)\,\Gamma(-\mu)\,\Gamma(\mu+2\nu+\kappa+2\omega)}\,\delta_{s,o} \tag{2.16}$$

which is Alekseev's[33] result (up to a phase when ν = integer).

3. PRINCIPLE VALUE PRESCRIPTION FOR THE AXIAL GAUGE SINGULARITY

In the last section the axial gauge singularity, together with the ultraviolet and infrared divergencies, were analytically regularized; the regularized S-integral is proportional to a G-function. In this section we shall show that when the axial gauge singularity is treated with the principal value prescription, the resulting S-integral can be expressed as the limit of a polynomial in a differential operator operating on a series of G-functions, and furthermore, that this result is equivalent to the earlier one.

According to the principle value prescription[12] the axial gauge singularity ($N \geq 1$) is regularized by the limiting process

$$\left(\frac{1}{q\cdot n}\right)^{2N+s} \to \frac{1}{2}\lim_{\eta\to o}\left[\left(\frac{1}{q\cdot n+i\eta}\right)^{2N+s} + \left(\frac{1}{q\cdot n-i\eta}\right)^{2N+s}\right] \tag{3.1a}$$

$$= \lim_{\eta\to 0}\left[\frac{1}{(q\cdot n)^2+\eta^2}\right]^{2N+s} \sum_{\ell=0}^{N} (q\cdot n)^{2N-2\ell+s}(-\eta^2)^{\ell}\binom{2N+s}{2\ell} \tag{3.1b}$$

$$= \lim_{\eta\to 0} R(\eta^2)\left[\frac{1}{(q\cdot n)^2+\eta^2}\right]^{N+s}(q\cdot n)^s \tag{3.1c}$$

where $s = 0$ or 1 and $R(\eta^2)$ is a differential operator in $\eta^2\,\partial/\partial\eta^2$ to be identified shortly. The regularization of the axial gauge singularity of an S-integral using this prescription thus involves the analysis of

$$S(p,n) = \lim_{\eta^2\to 0} R(\eta^2)T(p,n,\eta)$$

$$= \lim_{\eta^2\to 0} R(\eta^2) \int d^{2\omega}q(q^2)^{\mu}(q\cdot n)^{s}\left((p-q)^2\right)^{\kappa}\left((q\cdot n)^2 + \eta^2\right)^{\nu}, \tag{3.2}$$

where ν can be taken to be an integer if desired.

The procedure of Appendix A is exactly applicable to the integral $T(p,n,\eta)$ save the term in square brackets in (A.7) becomes

$$[\xi+\tau y(1-\xi)]^{\zeta} \to (D_y)^{\zeta}\left[1 + \frac{\eta^2\tau}{(1-\tau)(1-\xi)D_y}\right]^{\zeta} \tag{3.3}$$

in the notation of (B.2) and (A.9). Rewrite the binomial in (3.3) as a contour integral, a technique first employed by Capper and Leibbrandt[34], and transpose with the double integral of (A.7). We find

$$T(p,n,\eta^2) = \frac{(\pi)^{\omega}(p^2n^2)^{\nu}(p\cdot n)^{s}(p^2)^{\omega+\mu+\kappa}}{\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\kappa)}$$

$$\times \frac{1}{2\pi i}\int_L dt\, \frac{\Gamma(-t)\Gamma(\nu+s+1/2-t)(\eta^2/p^2n^2)^{t}}{\Gamma(2\omega+\kappa+\mu+2\nu-2t+s)} G^{2,3}_{3,3}\left(y\,\middle|\,\begin{matrix}t+1-s-\mu-\nu-\omega,\ \omega+1-t+\mu+\nu+\kappa,1+\nu-t;\\ 0,\ \omega+\kappa+\nu-t;\ 1/2-s\end{matrix}\right) \tag{3.4}$$

using (B.5). The contour L stretches from $-i\infty$ to $+i\infty$ enclosing the poles of $\Gamma(\nu+1/2+s-t)$ on the right because of (B.6a). As $\eta^2\to 0$ this displays a series of singularities of the form $(\eta^2)^{\nu+1/2+s}$, ... if $\mathrm{Re}(\nu) < -1/2-s$. These singularities may be removed by operating on T with

$$R(\eta^2) = \frac{\Gamma(\nu+1/2+s)}{\Gamma(\nu+N+1/2+s)}\left(s+\nu+1/2-\eta^2\frac{\partial}{\partial\eta^2}\right)\cdots\left(s+\nu+N-1/2-\eta^2\frac{\partial}{\partial\eta^2}\right) \tag{3.5}$$

where N+s = [−ν] ([] means greatest integer less than or equal to). This isolates the desired behaviour arising from the pole of $\Gamma(-t)$ at $t=0$. Now set $\nu = -N-s$ and identify

$$R(\eta^2) = \frac{\Gamma(-N+1/2)}{\Gamma(1/2)} \prod_{\ell=0}^{N-1} (-N+1/2+\ell-\eta^2 \frac{\partial}{\partial\eta^2}). \tag{3.6}$$

That this operator is equal to the regulator of (3.1c) is verifiable by proving the consequential result

$$\frac{\Gamma(1/2-N)}{\Gamma(1/2)} \prod_{\ell=0}^{N-1} \left(1/2-N+\ell-\eta^2 \frac{\partial}{\partial\eta^2}\right) \frac{(q\cdot n)^s}{((q\cdot n)^2+\eta^2)^{N+s}}$$

$$= \left(\frac{q\cdot n}{(q\cdot n)^2+\eta^2}\right)^{2N+s} \sum_{\ell=0}^{N} \left(-\eta^2/(q\cdot n)^2\right)^{\ell} \binom{2N+s}{2\ell} . \tag{3.7}$$

Expand the factor $((q\cdot n)^2 + \eta^2)^{-N-s}$ in the left hand side of (3.7) as an infinite series in $\eta^2/(q\cdot n)^2$, to obtain

$$(q\cdot n)^s\ R(\eta^2)((q\cdot n)^2+\eta^2)^{-N-s}$$

$$= (-)^N\ (q\cdot n)^{-2N-s}\ \frac{\Gamma(1/2-N)}{\Gamma(1/2)} \sum_k \frac{\Gamma(N+s+k)\,\Gamma(N+1/2+k)\left(-\eta^2/(q\cdot n)^2\right)^k}{\Gamma(N+s)\ \Gamma(1/2+k)\qquad k!} \tag{3.8a}$$

$$= \left(\frac{q\cdot n}{(q\cdot n)^2+\eta^2}\right)^{2N+s}\ {}_2F_1\left(-N,1/2-N-s;1/2;-\eta^2/(q\cdot n)^2\right) . \tag{3.8b}$$

The result (3.8b) is obtained by identifying the series in (3.8a) as a hypergeometric function, and applying Euler's transformation[35] to arrive at a truncated series. The right hand side of (3.7) is easily cast into the form of (3.8b) whenever s=0 or s=1 by using the duplication and reflection properties of the gamma function.

With the operator R specified in (3.6), the regularization defined by (3.2) now reduces to the result (2.2) because of (3.4) - the two procedures are equivalent. The order of divergence in (3.4) as $\eta^2 \to 0$ is reflected in the pole structure of (2.2) as a function of ν - possible poles at negative integer values of $\nu+s+1/2$.

4. EXPONENT DERIVATIVES, LOOP EXPANSION AND RENORMALIZATION

It is well known that physical quantities in renormalizable perturbation theories are represented by polylogs - power series in $(p^2)^n \ln^j p^2$ - generated in loop integrals. We shall show that the same set of polylogs is generated by exponent derivatives (including the zeroth derivative) of S-integrals (1.3), that these derivatives provide a useful basis for the solutions of integro-differential equations of nonperturbation theories, as in (1.2), and that such solutions are renormalizable.

4.1 Exponent derivatives and overlapping divergencies

In this section we discuss the regularization of S-integrals with logarithmic factors. This is important, as infinite parts which carry a logarithmic dependence on external momenta - we shall call these "logarithmic infinite parts" - cannot be renormalized by counterterms composed of local operators[5)]. In perturbation theories, such unrenormalizable singularities appear in "overlapping divergencies" of multi-loop integrals. It was shown by 'tHooft and Veltman[25)] that overlapping divergencies in a two-loop integral can be cancelled by subtracting from it counterterm insertions into the appropriate one-loop integrals. Here we shall first point out the close relation between the exponent derivative and the prescription of 'tHooft and Veltman.

Consider a two-loop diagram having the form

$$\int d^{2\omega}q\ K_1(q) \int d^{2\omega}q'\ K_2(q,q'), \qquad (4.1)$$

where references to external variables in the kernels K_1 and K_2 have been suppressed. Integration over q' with the result of Sec. 2 can be seen to yield

$$\int d^{2\omega}q\ K_1(q)\ (q^2)^{\varepsilon}\ \left(\frac{I}{\varepsilon} + R\right) \tag{4.2}$$

where I and R are functions in scalars composed of q and external variables; R may contain a factor of ℓny, cf. (4.11). For this discussion these factors are uninteresting and we set I=1 and R=0. The term of $O(1/\varepsilon)$, upon integration over q, generates a logarithmic infinite part. However, this singularity may be cancelled by an integral corresponding to the insertion of a counterterm into the appropriate one-loop integral, such that the combination reads

$$\int d^{2\omega}q\ K_1(q)\left[\frac{(q^2)^{\varepsilon}}{\varepsilon} - \frac{1}{\varepsilon}\right], \tag{4.3}$$

and is free of logarithmic infinite parts. This is the 'tHooft-Veltman prescription for dealing with overlapping divergencies. In Appendix D we demonstrate that this prescription removes overlapping divergencies generated in all multi-loop integrals. Now in the limit $\varepsilon \to 0$, the expression in the square parenthesis of (4.3) is suggestive of an exponent derivative:

$$\lim_{\sigma\to 0} \frac{d}{d\sigma}\,(q^2)^{\sigma} = \lim_{\varepsilon\to 0}\left[\frac{(q^2)^{\varepsilon}}{\varepsilon} - \frac{1}{\varepsilon}\right] = \ell n\ q^2. \tag{4.4}$$

Define a loop expansion as one such that the N^{th} level of the expansion contains all Feynman diagrams with N loops. At each loop level the 'tHooft-Veltman subtraction must be carried out to control the overlapping divergencies. Eq. (4.4) then suggests that, corresponding to this loop expansion in the diagrammatic approach for a perturbation theory, the analog for a nonperturbation theory has an expansion in a series of exponent derivatives. The common signature of both expansions is the power of logarithms in the calculated physical quantity.

In (4.4) a parameter distinct from ε was deliberately chosen for the exponent derivative. This was done to emphasize the notion that the 'tHooft-Veltman prescription corresponds to a special case of exponent derivatives with the limit $\sigma \to \varepsilon$. When the primal S-integrals in a nonperturbation theory contain more than one exponent, differentiation with respect to ε is not sufficient to deal with all distinct types of logarithms that are generated in the theory; the more general method of exponent derivatives must be developed.

4.2 Pole and logarithmic structure of the S-integral

In this section we demonstrate that S-integrals have at most simple poles, that exponent derivatives of order j have at most poles of order j+1 and that no logarithmic infinite parts need appear. For convenience we repeat the definition for the integral parts of the indices (2.5),

$$\begin{aligned} A_0 &= -M-N-s-2, \\ A_1 &= K+M+N+2, \\ A_2 &= N, \\ B_1 &= K+N+2; \end{aligned} \tag{4.5}$$

and their corresponding epsilons,

$$
\begin{aligned}
\varepsilon_0 &= \varepsilon+\sigma+\tau, \\
\varepsilon_1 &= -(\varepsilon+\rho+\sigma+\tau), \\
\varepsilon_2 &= -\tau, \\
\varepsilon_3 &= \varepsilon_b+\varepsilon_2 = \varepsilon+\rho.
\end{aligned} \tag{4.6}
$$

From the general discussion in Section 2, it is verifiable that the poles and logarithmic terms (we leave out the uninteresting factor π^{ω}) of the S-integral are all contained in the expression

$$
S \sim (p^2)^{-\varepsilon_1}(n^2)^{-\varepsilon_2}(\varepsilon_0+\varepsilon_b)^{k_0}(\varepsilon_1+\varepsilon_b)^{k_1}\varepsilon_2^{\,j_2}(\varepsilon_0+\varepsilon_1)^{h}
$$

$$
\times \Big\{\theta(B_1)\Big[\frac{P_1}{\pi_i\,\varepsilon_i^{j'_i}} + \frac{P_2}{\varepsilon_b}\Big(\frac{1}{\pi_i\,\varepsilon_i^{k'_i}} - \frac{y^{\varepsilon_b}}{\pi_i(\varepsilon_i+\varepsilon_b)^{k'_i}}\Big)\Big]
$$

$$
+\ \theta(-B_1-1)\Big[\frac{P_3}{\pi_i(\varepsilon_i+\varepsilon_b)^{k''_i}} + \frac{P_4}{\varepsilon_b}\Big(\frac{1}{\pi_i\,\varepsilon_i^{j''_i}} - \frac{y^{\varepsilon_b}}{\pi_i(\varepsilon_i+\varepsilon_b)^{j''_i}}\Big)\Big]\Big\} \tag{4.7a}
$$

where the summation is over $i=0,1,2$; all j's, k's and h are either 0 or 1, constrained by

$$
j_i = \theta(A_i) \geq \{j'_i,\ j''_i\} \geq 0, \qquad i = 0,1,2 \tag{4.7b}
$$

$$
k_i = \theta(A_i-B_1) \geq \{k'_i,\ k''_i\} \geq 0, \qquad i = 0,1,2 \tag{4.7c}
$$

$$
h = \theta(A_0 + A_1 + s); \tag{4.7d}
$$

$$
\begin{aligned}
\theta(x) &= 1 \qquad \text{if } x \geq 0, \\
&= 0 \qquad \text{if } x < 0,
\end{aligned} \tag{4.7e}
$$

P_1 and P_3 are terminating polynomials in y, P_2 and P_4 are regular functions of y, and all P's may be represented by non-singular series in the ε's. Of particular interest are the exponents $k_1 = \theta(M)$, $j_2 = \theta(N)$ and $h = \theta(K)$, which originate from the exponentiation of the three factors in the S-integral as in (A.1).

The regularization of the axial gauge singularity is transparently displayed in (4.7). The right-hand-side of (4.7a) may have a factor of $\frac{1}{\varepsilon_2} \sim \frac{1}{\tau}$ when any one of j_2', j_2'' or k_2' is equal to 1. Recall that ε_2 is independent of ε, signifying it is decoupled from dimensional regularization. Now if either j_2' or j_2'' or both are equal to unity, then (4.7b) ensures that $j_2 = 1$, so that the factor $1/\tau$ is cancelled by the factor $\varepsilon_2^{j_2} \sim \tau$ in the numerator. If $k_2' = 1$, then from (4.7a) and (4.7c), $B_1 \geq 0$ and $A_2 - B_1 \geq 0$, so that $A_2 \geq 0$. Therefore $j_2 = 1$ again and the factor $1/\tau$ is cancelled. This demonstrates that S is free of axial gauge singularities. In general, the θ-functions in (4.7a) imply the additional inequalities

$$k_i \geq j_i'', \quad i = 0,1, \tag{4.7f}$$

$$j_2 \geq k_2'. \tag{4.7g}$$

Using arguments similar to those given above, we prove the following two theorems. As a rule the order of operation is: set $\rho=\sigma=\tau=0$, while ε is kept finite but small. We shall refer to each term in (4.7a) by the function P_i multiplying it.

Theorem 1. In the limit ρ, σ and $\tau \to 0$, S can be at most as singular as $1/\varepsilon$.

The P_1 term is most singular when $j_0' = j_1' = 1$ and is $O(\frac{1}{\varepsilon})$ by virtue of (4.7d). The P_3 term is $O(\frac{1}{\varepsilon})$ when $k_2'' = 1$; it is regular, by virtue of

(4.7c), when $k_0'' = 1$ or $k_1'' = 1$, or both. The $P_2(P_4)$ term is at most as singular as the $P_3(P_1)$ term because the difference between its terms is proportional to ε_b cancelling the factor of ε_b in the denominator. The theorem can alternatively be proven using the general discussion of Section 2 and the observation from Figs. 1 and 2 that there are no double pinches without at least one compensating zero.

Theorem 2. Terms of $O(1/\varepsilon)$ in S are independent of $\ell n p^2$, $\ell n(n^2)$ or $\ell n y$; all logarithmic terms are of O(1) or of higher order in ε. This follows simply from Theorem 1 and from the fact that all logarithmic terms devolve from the factors $(p^2)^{-\varepsilon_1}$, $(n^2)^{-\varepsilon_2}$ and y^{ε_b}. These two theorems possibly have been proven before. The proofs are trivial for covariant gauge integrals, i.e., when $N = s = 0$.

We now discuss the pole and logarithmic structure of exponent derivatives. The order of limits is: first differentiate, then set $\rho=\sigma=\tau=0$; ε is kept finite. Consider the three types of "two-loop" integrals:

$$\int d^{2\omega}q\cdots\ell n(p-q)^2 = \lim_{\rho\to 0}\frac{d}{d\rho}\,S(K+\rho,M,N,s), \tag{4.8a}$$

$$\int d^{2\omega}q\cdots\ell n q^2 = \lim_{\sigma\to 0}\frac{d}{d\sigma}\,S(K,M+\sigma,N,s), \tag{4.8b}$$

$$\int d^{2\omega}q\cdots\ell n[(q\cdot n)^2/(q^2n^2)] = \lim_{\substack{\sigma\to 0\\ \tau\to 0}}\left(\frac{d}{d\tau}-\frac{d}{d\sigma}-\ell n n^2\right)S(K,M+\sigma,N+\tau,s) \tag{4.8c}$$

where $\cdots = [(p-q)^2]^K(q^2)^M(q\cdot n)^{2N+s}$ represents the integrand of the primal S-integral. The discussion of the exponent derivatives is simplified if one defines $\tilde{S}$ by

$$S = (n^2)^\tau \tilde{S}, \tag{4.9}$$

so that

$$\left(\frac{d}{d\tau} - \frac{d}{d\sigma} - \ell n n^2\right) S = (n^2)^\tau \left(\frac{d}{d\tau} - \frac{d}{d\sigma}\right)\tilde{S}. \tag{4.10}$$

From (2.2), it is seen that as far as exponent derivatives are concerned, $\tilde{S}$ is independent of n^2 and has the structure

$$\tilde{S} = (p^2)^{-\varepsilon_1}(\text{pole terms + regular terms}).$$

Henceforth in this section by S we mean $\tilde{S}$; the term $\ell n n^2$ in (4.8c) and the factor $(n^2)^\tau$ in (4.10) will be ignored. From (4.8), it is clear that integrals with higher powers of logarithms in the integrand are simply related to the appropriate exponent derivatives of S.

From (4.7), with nonessential terms and factors suppressed,

$$S \sim (p^2)^{-\varepsilon_1}\left[\frac{a_0}{\varepsilon_0} + \frac{a_1}{\varepsilon_1} + \frac{a_3}{\varepsilon_3} + \frac{c}{\varepsilon_b}(1-y^{\varepsilon_b})\right] \tag{4.11}$$

where a_0, a_1, a_3 and c are polynomials in the epsilons. The last term in the square parenthesis above is actually finite; it is written out explicitly to demonstrate that the factor y^{ε_b}, peculiar to the axial gauge, never generates logarithmic infinite terms. From now on we ignore this term by equating c to zero.

The small variable ε_1 is of particular interest: it is associated with the ultraviolet divergence of the integral and contains all the small variables ρ, σ, τ and ε. It will be shown shortly that gauge-independent high order logarithms (i.e. power of $\ell n p^2$ but not of $\ell n y$) are generated only by derivatives with respect to ε_1. We therefore re-express the poles of $O(1/\varepsilon_0)$ and $O(1/\varepsilon_3)$ in the form

$$\frac{1}{\varepsilon_i} = -\frac{1}{\varepsilon_1} + \left(\frac{1}{\varepsilon_i} + \frac{1}{\varepsilon_1}\right) = -\frac{1}{\varepsilon_1} - \frac{\sigma_i}{\varepsilon_i \varepsilon_1}, \quad i = 0,3, \tag{4.12}$$

where $\sigma_0 \equiv \rho$ and $\sigma_3 \equiv \sigma + \tau$. Note that for $i=0,3$, ε_i is independent of σ_i:

$$\frac{d}{d\sigma_i} \varepsilon_i = 0, \qquad i = 0,3. \tag{4.13}$$

The poles and associated logarithmic terms of S thus devolve from

$$S \sim \frac{(p^2)^{-\varepsilon_1}}{(-\varepsilon_1)} \left(a_1 + \sum_{i=0,3} a_{2i} \frac{\sigma_i}{\varepsilon_i}\right), \tag{4.14}$$

where a_1, a_{2i} are regular functions of the epsilons with leading O(1) terms. We shall refer to the summation in the parenthesis as the "a_2-terms". In the limit $\rho=\sigma=\tau=0$, these terms are zero and may therefore always be deleted (cf. Section 2). They generate logarithmic infinite parts when S is differentiated with respect to ρ, σ or τ.

Without loss of generality, consider S operated upon with $(d/d\sigma)^j$. From (4.14),

$$\lim_{\rho,\sigma,\tau\to o} \left(\frac{d}{d\sigma}\right)^j S = \lim_{\sigma\to o} \sum_{r=o}^{j} \binom{j}{r} \left[\left(\frac{d}{d\varepsilon}\right)^{j-r} \frac{(p^2)^{\varepsilon}}{\varepsilon}\right]\left[\left(\frac{d}{d\sigma}\right)^r \left(a_1 + \frac{\sigma}{\varepsilon} a_2\right)\right], \tag{4.15}$$

where

$$\lim_{\sigma\to o} \left(\frac{d}{d\sigma}\right)^r a_1(\varepsilon,\sigma) \equiv a_1^{(r)} \sim O(1), \tag{4.16a}$$

$$\lim_{\sigma\to o} \left(\frac{d}{d\sigma}\right)^r \frac{\sigma}{\varepsilon} a_2(\varepsilon,\sigma) \equiv \frac{1}{\varepsilon}\, a_2^{(r-1)} \sim O\!\left(\frac{1}{\varepsilon}\right), \tag{4.16b}$$

$$\left(\frac{d}{d\varepsilon}\right)^r \frac{(p^2)^{\varepsilon}}{\varepsilon} = \frac{(-)^r r!}{\varepsilon^{r+1}} + \frac{1}{r+1}\, \ln^{r+1}(p^2) + O(\varepsilon). \tag{4.16c}$$

By virtue of the identity (4.16c) (for a proof, expand in ε and use (D.7)), one sees that the right-hand-side of (4.15) has the structure

$$\sum_{r=0}^{j} a_1^{(r)} \left[O(\frac{1}{\varepsilon^{j-r+1}}) + \ell n^{j-r+1}(p^2) O(1) \right] + \sum_{r=1}^{j} a_2^{(r-1)} O(\frac{1}{\varepsilon^{j-r+2}})$$

$$+ \sum_{r=1}^{j} a_2^{(r-1)} \ell n^{j-r+1}(p^2) O(\frac{1}{\varepsilon}). \quad (4.17)$$

Examination of derivatives with respect to ρ and τ can be carried out along the same lines and will not be shown here. The result can be summarized in the following theorems.

Theorem 3. An exponent derivative, of any order, of an S-integral that is finite is also finite. In (4.14) both a_1 and a_2 do not appear and no singularities can be generated by differentiating the remaining regular terms.

Theorem 4. The j^{th}-order exponent derivative of an S-integral that has infinite terms of $O(\frac{1}{\varepsilon})$, but no terms of $O(\frac{\rho}{\varepsilon^2})$, $O(\frac{\sigma}{\varepsilon^2})$ or $O(\frac{\tau}{\varepsilon^2})$ contains infinite terms that are as singular as $O(\frac{1}{\varepsilon^{j+1}})$ and finite terms that have logarithms up to the (j+1) power. No logarithmic infinite terms are generated. This is verified by considering (4.17) in the absence of a_2-terms.

Theorem 5. If an S-integral has infinite terms of $O(\frac{\rho}{\varepsilon^2})$, $O(\frac{\sigma}{\varepsilon^2})$ or $O(\frac{\tau}{\varepsilon^2})$ then any ρ, σ or τ derivative of it will generate logarithmic infinite terms. This can be seen in (4.17) when a_2 and its derivatives are non-zero.

The proofs of Theorems 1-5 are strictly based on our study of the S-integrals, which by no means represent every type of Feynman integral that may appear in the perturbation expansion of a field theory. In particular our S-integrals contain at most only one external momentum. Without providing a rigorous proof that the theorems are also true for integrals with any number of external momenta (for a renormalizable field theory, only integrals with a maximum of three external momenta need be considered) we sketch in the following how one could proceed to establish such a proof.

Consider a primal integral with an integrand containing a product of m (scalar) factors, each being a power of a quadratic expression in external momenta p_k and/or q, and of $q \cdot n$, in the case of an axial gauge. The notation is as before. Each of the factors is then analytically regulated by first making its exponent continuous: $K_j \to \kappa_j$, then expressing it in integral form by exponentiation as in (A.1). The integration over the 2ω-dimensional q-space is then carried out (cf. (A.4)). The scale parameter can also be integrated out, reducing the integral to the form

$$S_{2\omega}(\{p_k\},n;\{K_j\}) \sim (p^2)^{\alpha_1}\,\Gamma(-\alpha_1)\Big[\prod_{j=1}^{m}\Gamma(-\kappa_j)\Big]^{-1}\ (m-1) \tag{4.18}$$

where $\alpha_1 = \omega + \sum_i \kappa_j$ is the ultraviolet index with associated epsilon ε_1, the factor $(p^2)^{\alpha_1}$ represents the dimension of the integral, and $(m-1)$ is a "canonical" (m-1)-fold integral having the form (cf. (A.7))

$$(m-1) = \Big(\prod_{j=2}^{m}\int_0^1 d\xi_j\Big)\prod_{\ell=2} f_\ell^{\,-\alpha_\ell-1} \tag{4.19}$$

where each factor f_ℓ is a function of the integration parameters ξ_j and dimensionless scalar products of p_k and n; the indices α_ℓ will be linear combinations of κ_j and ω. No α_ℓ other than the ultraviolet index α_1 depends on ω and all of the κ_j's. Depending on the magnitude of m, the (m-1)-fold integral on the right-hand-side of (4.19) may be a hypergeometric function, or a generalization or transcendent of it, or a sum of such functions; the S-integral of (1.1) in the axial gauge corresponds to m=3. The important point is that each parametric integration in (4.19) may effectively induce poles from end point singularities which have as arguments linear combinations of α_ℓ, and therefore of ω and κ_j. The final form for (4.18) may be a sum of terms, each being a product of poles and zeros. At any point in the ω and κ_j space of m+1 dimensions, the poles will be up to order m+1 and the zeros up to order m. Consider the behavior of the integral in the region where the exponents and ω have close to integral values, namely $\kappa_j = K_j+\rho_j$, K_j = integer, $\omega = 2+\varepsilon$, ρ_j and ε small. The above argument suggests that in the limit $\rho_i \to 0$ and $\varepsilon \to 0$, the poles and zeros of the gamma functions conspire in such a way that the most divergent terms of the integral are simple poles of the type

$$S \sim (p^2)^{-\varepsilon_1}\left(\sum_i \frac{a_i}{\varepsilon_i}\right) \tag{4.20}$$

where each ε_i is ε plus a distinct linear combination of ρ_j's with all coefficients being unity,

$$-\varepsilon_1 = \varepsilon + \sum_{j=1}^{m} \rho_j \tag{4.21}$$

is the epsilon associated with the ultraviolet index, and a_i are regular functions of ε and ρ_j with leading terms of $O(1)$. We have not yet succeeded in proving this, although it may be that the proof is known; (4.20) certainly has the correct $\rho_j=0$ limit for all Feynman integrals usually encountered in the literature. It is important to note that, just as in (4.11), the dimension of the integral dictates that the exponent of p^2 in (4.20) must be $-\varepsilon_1$, whereas the poles can be generated by a number of ε_i's other than ε_1.

Rewrite poles in ε_i, $i\neq 1$ using partial fractions:

$$\frac{1}{\varepsilon_i} = -\frac{1}{\varepsilon_1} + \frac{\delta\varepsilon_i}{\varepsilon_i\,\varepsilon_1} \tag{4.22}$$

where $\delta\varepsilon_i \equiv \varepsilon_1+\varepsilon_i$ is a linear combination of ρ_j's. Then

$$S \sim \frac{(p^2)^{-\varepsilon_1}}{(-\varepsilon_1)} \left(a_1' - \sum_{i>1} a_i \frac{\delta\varepsilon_i}{\varepsilon_i}\right), \tag{4.23}$$

which is of the form of (4.14). The summation in the parenthesis will again be refered to as the a_2-term; in the limit $\rho_j\to 0$ it is of $O(\rho_j/\varepsilon^2)$. It follows from Theorems 4 and 5 that in the $\rho_j\to 0$ limit, exponent derivatives of S contains high order poles but are free of logarithmic infinite parts if and only if the a_2-term is absent.

4.3 Regularization by Subtraction

In this section we propose a method of evaluating exponent derivatives free from logarithmic infinite parts. The a_2-terms appearing in (4.14) and (4.23) which generate logarithmic infinite parts are artifacts of regularization. As an example, consider the sum of poles

$$A \equiv \frac{1}{-\varepsilon_1} + \frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3} \qquad (4.24)$$

having different analytical properties under differing limits

$$\lim_{(\rho,\sigma,\tau)\to 0} A = \frac{3}{\varepsilon}, \qquad (4.25a)$$

$$\lim_{(\varepsilon,\rho,\sigma)\to 0} A = \frac{2}{\tau} + \text{indefinite term}, \qquad (4.25b)$$

$$\lim_{(\varepsilon,\sigma,\tau)\to 0} A = \frac{2}{\rho} + \text{indefinite term}, \qquad (4.25c)$$

$$\lim_{(\varepsilon,\rho,\tau)\to 0} A = \frac{2}{\sigma} + \text{indefinite term}. \qquad (4.25d)$$

This example illustrates one of the ambiguities of a singular function generalized from a set of integers. The ambiguity can therefore be removed, or regulated, arbitrarily. We choose to regulate by first re-expressing all poles, using partial fractions, as poles in ε_1, and then discarding the remainder (see (4.12) and (4.22)), which normally vanishes anyway when ρ, σ and τ are zero. This amounts to subtracting all a_2-terms in (4.14) and (4.23), after which the S-integral consists only of regular parts and $1/\varepsilon_1$ poles. These can be evaluated independently of the order of the limiting process and, by virtue of Theorem 4, are free from logarithmic infinite terms. The equality

$$\lim_{(\rho,\sigma,\tau)\to 0} \frac{d}{d\rho_i}\left[\frac{(p^2)^{-\varepsilon_1}}{(-\varepsilon_1)}\right] = \frac{d}{d\varepsilon}\left[\frac{(p^2)^{\varepsilon}}{\varepsilon}\right] \qquad \rho_i = \rho,\ \sigma \text{ or } \tau \quad (4.26)$$

asserts that, as far as high order poles and associated logarithms of the S-integral (without the factor π^{ω}) are concerned, all exponent derivatives are equivalent to derivatives with respect to ε (ε-derivatives), and all such high order poles are generated by the ultraviolet divergence.

In summary, if the expansion series in the epsilons for S is

$$S = (p^2)^{-\varepsilon_1}\left(\frac{I_1}{-\varepsilon_1} + \sum_{i\neq 1} \frac{I_i}{\varepsilon_i}\right) + R, \tag{4.27a}$$

where for the regular part the factor $(p^2)^{-\varepsilon_1}$ has been absorbed into R, then the regulated S, for exponent derivatives, is

$$S_{reg} = \frac{(p^2)^{-\varepsilon_1}}{(-\varepsilon_1)}\ I + R, \qquad I \equiv \sum_i I_i, \tag{4.27b}$$

and the exponent derivative corresponding to the operator

$$D^j \equiv \prod_i \left(\frac{d}{d\kappa_i}\right)^{j_i} = \prod_i \left(\frac{d}{d\rho_i}\right)^{j_i}, \quad j \equiv \sum_i j_i \tag{4.27c}$$

is

$$\lim_{\{\rho_i\}\to 0} D^j S_{reg} = \lim_{\{\rho_i\}\to 0} \left\{\left[\left(\frac{d}{d\varepsilon}\right)^j \frac{(p^2)^{\varepsilon}}{\varepsilon}\right] I + \frac{(p^2)^{\varepsilon}}{\varepsilon} D^j I + D^j R\right\}. \tag{4.27d}$$

That is, all poles ε_0^{-1}, ε_1^{-1} and/or ε_3^{-1} in S have become $-\varepsilon_1^{-1}$ in S_{reg}.

We mention in passing that an alternative regularization to the one discussed here, where ultraviolet and infrared poles are treated indiscriminately, is to isolate the two types of poles. The rationale for this is that only ultraviolet divergencies need be renormalized, while infrared divergencies presumably would automatically cancel in the probability of any physical process, provided that soft gauge-particles emitted in the final state are taken into account. In this alternative, infrared, but not ultraviolet, poles will have logarithmic residues in exponent derivatives.

4.4 Renormalization by Subtraction

We make a few remarks on the arbitrariness in the final step of the regularization process - the removal of ultraviolet divergencies - commonly referred to as "renormalization"[5]. In dimensional regularization, the simplest procedure (the minimal subtraction scheme (MS)),[36] is to subtract from the integral all poles and nothing else. The implication is that a Lagrangian composed of counterterms can be constructed which generates (infinite) terms that exactly cancel the poles and nothing else. In the limit $(\rho,\sigma,\tau)\to o$, poles of the S-integral always appear in the combination

$$1/\varepsilon + \ln\pi + \gamma. \tag{4.28a}$$

The $\ln\pi$ term arises from the factor π^{ω} in the S-integral and the Euler constant γ is associated with the residue of the pole. Sometimes Feynman integrals are defined such that they have an extra factor of normalization $(2\pi)^{-2\omega}$ in (2.1), in which case (4.28a) becomes

$$1/\varepsilon - \ln 4\pi + \gamma. \tag{4.28b}$$

The renormalization procedure whereby the $O(\varepsilon^{-1})$ term is subtracted in the combination (4.28b) is known as the $\overline{\mathrm{MS}}$ scheme[37].

The term $\ln\pi$ (or $\ln 4\pi$) is an artifact of the regularization. Had we chosen to use the method of analytic regularization exclusively (i.e. set $\varepsilon=0$ at the outset so that the infinite parts are of $O(\sigma^{-1})$ or $O(\rho^{-1})$ - see section 2.1 for detail), then the term $\ln\pi$ would never have appeared. In this sense it is not only legitimate, but indeed natural, to remove it by subtraction.

In our method of regularization, the usefulness of generalizing to continuous dimensions chiefly resides in the expeditious evaluation of the simultaneous limit $(\rho, \sigma, \tau)\to 0$; ω serves a useful purpose only when it appears in linear combination with other exponents in one of the indices (α_i), but not when it appears singly as in the factor π^{ω}. This implies that, among other things, dimensional regularization need not be extended to the domain of Dirac algebra. In practice it means that dimensional regularization can, and in our view should, be implemented only <u>after</u> the algebra has been done in four dimensional space-time. Compliance with this procedure would remove a contentious ambiguity associated with the conventional method of dimensional regularization: that of a consistent definition of antisymmetric tensors in a space of continuous dimensions. A serious consequence of this ambiguity is that it generates spurious γ_5-anomalies[3,15,25]. This procedure is similar to that of the recently proposed method of dimensional reduction[17]. However, contrary to arguments given there, the discussion in Sec. 2.1 shows that the condition $\omega \leq 2$ need not be imposed (cf. (2.3)).

The point of view that only integrals need be regulated also sets our method of analytic regularization apart from that of Speer[14]. Because Speer regulates propagators, his method appears not to preserve gauge invariance[3]. Our method is free from such criticism because it was earlier shown that with it the regulated integrals are equivalent to those obtained with dimensional regularization, which is known to preserve gauge invariance.

4.5 Renormalizability of Perturbation and Nonperturbation Theories

In perturbation theory physical amplitudes are represented by Feynman integrals. We have shown that taking the exponent derivative of N-loop Feynman integrals is closely related to computing (N+1)-loop integrals and then subtracting from it counterterm insertions into N-loop integrals, according to the 'tHooft-Veltman scheme. The pole and associated logarithmic structure of (N+1)-loop integrals is therefore similar in both prescriptions. Perturbation theories according to the 'tHooft-Veltman prescription are therefore renormalizable by the standard subtraction method.

Now consider a nonperturbation calculation such as that induced by the Schwinger-Dyson equation (1.2) for the gluon propagator. One may possibly solve the equation by iteration: start with a zeroth order propagator $Z^{(0)}$ without logarithm. The integration will generate terms with one power of logarithm, plus divergent terms which are subtracted. This completes one iteration. Now substitute the new $Z^{(1)}$, including logarithms, back into the integral. This generates terms with two powers of logarithms, and so on. Formally, this iteration procedure generates a continuous fraction representation for Z,

$$Z^{-1} = 1 + H(Z)Z, \tag{4.29a}$$

or

$$Z = \cfrac{1}{1 + H \cfrac{1}{1 + \ddots H \cfrac{1}{1+HZ^{(0)}}}} \tag{4.29b}$$

where in (4.29b) at each stage of the iteration the integral operator H is a function of Z obtained from the preceeding iteration.

Since logarithms are generated by the evaluation of S-integals and their exponent derivatives, an alternative to solving (4.29) by iteration, involving the evaluation of many-fold 2ω-dimensional integrals, is to construct a trial Z with logarithms, and solve (4.29a) directly by evaluating the one-fold integral HZ using the method of exponent derivatives. Restricting the trial solution to contain logarithms of up to the N^{th} power corresponds to computing Z to the N-loop level in perturbation theory. The integral HZ itself, where Z contains logarithms, can be generated by evaluating exponent derivatives of an S-integral. We have shown previously that S-integrals and their exponent derivatives can be regulated such that they contain at most only poles due to ultraviolet divergencies so that these can be renormalized by subtraction. This indicates that, in principle, a nonperturbative, renormalized solution of (4.29a), of any order corresponding to that of the same order in a loop expansion, can be calculated with the methods of dimensional and analytic regularization in conjunction with the application of exponent derivatives.

5. SOME EXAMPLES

We evaluate some special cases to illustrate the power of the G-function representation and illuminate some properties discussed generally in Sections 2 and 4. Following the discussion of Section 4.3, the factor π^{ω} is isolated when S is singular, and is written as π^2 when S is regular.

5.1 A Regular Integral

The case $\mu = \kappa = \nu = -1$, $s = 1$ is of some interest because $\omega=2$ lies within the window of convergence defined by (2.3); there is no analytic continuation except in y, nor any regularization. Consequently the integral will be well-defined in ω, κ, μ and ν and analytic in y. From (2.2) we have

$$S = \frac{\pi^2}{p^2 n^2} \Gamma(\tfrac{1}{2})(p\cdot n)\, G^{2,3}_{3,3}\left(y \,\middle|\, \begin{matrix} 0,0,0; \\ 0,0;-1/2 \end{matrix}\right) \tag{5.1}$$

and there is no necessity to take ε limits within the G-function because the contour is not pinched. From (2.8) and utilizing the residue theorem for a dipole we find

$$S = \frac{-\pi^2(p\cdot n)\Gamma(1/2)}{p^2 n^2} \sum_{\ell} \frac{\Gamma(1+\ell)}{\Gamma(\frac{3}{2}+\ell)} y^{\ell}[\psi(1+\ell) - \psi(\tfrac{3}{2}+\ell) + \ln y] \tag{5.2}$$

and the series converges for $|y| < 1$, albeit lackadaisically. An equivalent form for this integral has been given by van Neerven[38] (Appendix C).

The analytic continuation to $|y| > 1$ given in (2.12), leads to

$$S = \frac{\pi^2 \Gamma(1/2)}{p \cdot n} G^{3,2}_{3,3}\left(\frac{1}{y}\middle| \begin{matrix} 0,0;1/2 \\ 0,0,0; \end{matrix}\right)$$

$$= \frac{\pi^2}{2p \cdot n} \sum_{\ell} \frac{(-y)^{-\ell}}{\Gamma(\ell+1)\Gamma(\frac{1}{2}-\ell)} \{[\psi(\tfrac{1}{2}-\ell)-\psi(1+\ell)-\ell n\, y]^2 + 2\psi'(\tfrac{1}{2}) - \psi'(1+\ell) - \psi'(\tfrac{1}{2}-\ell)\}. \tag{5.3}$$

The series slowly converges for $|y| > 1$; ψ and ψ' are polygamma functions (digamma and trigamma respectively). As $1/y \to 0^+$, the leading behaviour (light-cone gauge) is

$$S(p,n) \sim \ell n^2 y \,, \tag{5.4}$$

demonstrating that the function $R(y,p,n)$ of (2.10) is singular at the point $n^2 = 0^+$, although the leading pole structure in Fig. 2 and the condition (2.13) for this example nominally implies $(1/y)^0$ behaviour.

This case also illustrates how the cut extending from $[1,\infty]$ in the y-plane vanishes. Retain the ε dependence and use (2.7) to write

$$S = \frac{\pi^{\omega} y}{p \cdot n} \frac{\Gamma(1/2)}{\Gamma(1+2\varepsilon)} \left[\frac{\Gamma(\varepsilon)\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3/2)\Gamma(1+2\varepsilon)} \,{}_2F_1\left(\begin{matrix}1,1+\varepsilon \\ 3/2\end{matrix}\middle| y\right)\right.$$

$$\left. + \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(3/2+\varepsilon)} y^{\varepsilon} \,{}_2F_1\left(\begin{matrix}1,1+2\varepsilon \\ 3/2+\varepsilon\end{matrix}\middle| y\right)\right] \tag{5.5}$$

Employ a linear transformation $y \to \frac{y-1}{y}$ to expand the hypergeometric functions about $y = 1$:

$$S = \frac{\pi^2 y \Gamma(1/2)\Gamma(1+\varepsilon)}{p\cdot n\ \Gamma(1+2\varepsilon)} \Big\{ y^{-1-\varepsilon} \Big(\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(-1/2-\varepsilon)}{\Gamma(1+2\varepsilon)\Gamma(1/2-\varepsilon)\Gamma(1/2)}\ {}_2F_1\Big(\begin{matrix}1+\varepsilon, 1/2+\varepsilon\\ 3/2+\varepsilon\end{matrix}; \frac{y-1}{y}\Big)$$

$$+ \frac{\Gamma(-\varepsilon)\Gamma(-1/2-\varepsilon)}{\Gamma(1/2-\varepsilon)\Gamma(1/2+\varepsilon)}\ {}_2F_1\Big(\begin{matrix}1+2\varepsilon, 1/2+\varepsilon\\ 3/2+\varepsilon\end{matrix}; \frac{y-1}{y}\Big)\Big)$$

$$+ y^{-1/2}(1-y)^{-1/2-\varepsilon} \frac{\Gamma(1/2+\varepsilon)}{\Gamma(1+2\varepsilon)} \Big[\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)} + \Gamma(-\varepsilon)\Big]\Big\} \qquad (5.6)$$

The factor in the square brackets vanishes identically for all ε, thereby eliminating the explicit cut which exists in each of the original hypergeometric functions independently. From the remaining terms, the value of the integral at this point may be obtained:

$$S(y=1) = \frac{2\pi^2 \ln 4}{(p\cdot n)} \quad . \qquad (5.7)$$

5.2 Covariant Gauge

Here we are interested in the case in which $\nu=s=0$; S reduces to

$$S(p;\kappa,\mu) \equiv \int d^{2\omega}q\ (q^2)^{\mu}\ [(p-q)^2]^{\kappa} \qquad (5.8)$$

and all n-dependence vanishes. From contiguity relations between G-functions[26], it is easy to extract a factor $1/\nu$ from the G-function in (2.2), write the sum of contiguous G-functions as a contour integral[30], and set $\nu=0$. We find that all residues vanish save the one at the origin and no limiting process is required. The result is[4]

$$S(p;\kappa,\mu) = \frac{\pi^{\omega}(p^2)^{\omega+\mu+\kappa}\ \Gamma(\omega+\kappa)\Gamma(\omega+\mu)\Gamma(-\omega-\mu-\kappa)}{\Gamma(-\mu)\Gamma(-\kappa)\Gamma(2\omega+\mu+\kappa)} \qquad (5.9)$$

Note the symmetry with respect to interchange of μ and κ, as expected from the "shift" property of the integral representation (5.8). In addition, because of the order of limits, it is obvious from (5.9) that

$$S(p;K,\mu) = 0 , \qquad K \geq 0, \tag{5.10a}$$

$$S(p;\kappa,M) = 0 \qquad M \geq 0. \tag{5.10b}$$

This confirms a conjecture relating to the properties of tadpole diagrams,[39]

$$S(p;0,M) = 0, \qquad M \geq 0. \tag{5.11}$$

Because of the conditions (2.3b,c) and reasons given in Sect. 2.2, (5.10, 11) cannot be derived with the method of dimensional regularization alone.

For a summary of all the nonzero cases in the covariant gauge, now let

$$\kappa = -K-1, \qquad K = 0,1, \cdots$$

$$\mu = -M-1, \qquad M = 0,1, \cdots \tag{5.12}$$

A straightforward analysis of (5.9) gives the following possibilities:

<u>M = K = 0</u>

$$S(p;-1,-1) = - \pi^{\omega} \left(\frac{1}{\varepsilon} + \gamma + \ell n p^2 - 2\right) , \tag{5.13a}$$

<u>M = 0, K = 1</u> or <u>M = 1, K = 0</u>

$$S(p;-1,-2) = S(p;-2,-1) = \frac{\pi^{\omega}}{p^2} \left(\frac{1}{\varepsilon} + \gamma + \ell n p^2\right) , \tag{5.13b}$$

<u>$M + K \geq 2$</u>

$$S(p;-K-1,-M-1) = 2\pi^{\omega}\,(p^2)^{-M-K}\;\frac{\Gamma(M+K)\,\Gamma(M+K-1)}{\Gamma(M+1)\,\Gamma(K+1)\,\Gamma(M)\,\Gamma(K)}$$

$$\times \left\{\frac{1}{\varepsilon} + \left[\ell n p^2 - \psi(M+K) - 2\psi(M+K-1) + \psi(M) + \psi(K)\right]\right\} . \qquad (5.13c)$$

The simplicity of the result (5.9) permits a generalization to the J-fold integral

$$S^J(p;\{\kappa_i\};\{\mu_i\}) = \int d^{2\omega_1}q_1\; S(p) \int d^{2\omega_2}q_2\; S(q_1) \cdots \int d^{2\omega_J}q_J S(q_{J-1})$$

$$\equiv \int d^{2\omega_1}q_1 (q_1^2)^{\mu_1}\left((q_1-p_1)^2\right)^{\kappa_1} \int d^{2\omega_2}q_2 (q_2^2)^{\mu_2}\left((q_2-q_1)^2\right)^{\kappa_2} \cdots$$

$$\times \int d^{2\omega_J}q_J (q_J^2)^{\mu_J}\left((q_J-q_{J-1})^2\right)^{\kappa_J}$$

$$= \pi^{2W_1}(p^2)^{M_1+K_1+W_1}$$

$$\times \prod_{i=1}^{J} \frac{\Gamma(\kappa_i+\omega)\,\Gamma(M_i+W_i+K_{i+1})\,\Gamma(M_i-K_i-W_i)}{\Gamma(-\kappa_i)\,\Gamma(-M_i-K_{i+1}-W_{i+1})\,\Gamma(M_i+K_i+W_i+\omega_i)} \qquad (5.14)$$

where

$$M_h = \sum_{i=h}^{J} \mu_i , \tag{5.15a}$$

$$K_h = \sum_{i=h}^{J} \kappa_i , \tag{5.15b}$$

$$W_h = \sum_{i=h}^{J} \omega_i \tag{5.15c}$$

and E_h, K_h and M_h are all zero if $h > J$. Note that (5.14) is a general result allowing all the ω's to be distinct, whereas the conventional application[3] of dimensional regularization is restricted to the special case $\omega_1 = \omega_2 = \cdots = \omega$ (cf. Appendix D).

5.3 Axial Gauge with $\kappa = -1$

Here we consider S-integrals of the form

$$X(i,j) \equiv S(p,n;-1,M,N-s,s)$$

$$= \int d^{2\omega}q(q^2)^M(q\cdot n)^{2N-s}\left((p-q)^2\right)^{-1} \tag{5.16}$$

with i=M, j=s-2N, some of which had their infinite parts originally calculated by Capper and Leibbrandt,[13] part of whose notation (X) we retain. In the following we have

$$M \geq 0 , \tag{5.17a}$$

$$N \leq s-1 . \tag{5.17b}$$

According to (2.5),

$$A_0 = -M-N-2,$$

$$A_1 = M+N-s+1,$$

$$A_2 = N-s \leq -1,$$

$$B_1 = 1+N-s \leq 0, \tag{5.17c}$$

so

$$B_1 - A_1 = -M \leq 0 \tag{5.17d}$$

and

$$A_0 + A_1 = -1-s < 0. \tag{5.17e}$$

This shows that for such integrals $B_1 \leq A_1$, and A_0 must be of opposite sign to A_1 except in one eventuality ($A_0 = A_1 = -1$, $s=1$). This limits the possible combinations of poles in the overlap region (cf. Section 2.3) to five. They are:

$$A_1 \geq 0 > A_0 \geq B_1 > A_2 \tag{5.18a}$$

$$A_0 \geq 0 > A_1 \geq B_1 > A_2 \tag{5.18b}$$

$$-1 = A_0 = A_1 \geq B_1 > A_2 \tag{5.18c}$$

$$A_1 \geq 0 > B_1 > A_2 \geq A_0 \tag{5.18d}$$

$$A_1 \geq 0 = B_1 > A_2 \geq A_0 \tag{5.18e}$$

Of these possibilities, only the last two contribute an infinite part to $X(i,j)$; the first three are symmetric under the interchange $A_1 \leftrightarrow A_0$ and are all regular.

The first three cases may be summarized by evaluating the residues:

$$X(M,s-2N) = -\frac{\pi^2 (p^2)^{A_1} (n^2)^{A_2} (p\cdot n)^s \Gamma(1/2+A_2+s)\Gamma(-B_1) y^{B_1}}{\Gamma(-A_2)\Gamma(1/2+B_1+s)} \tag{5.19}$$

$$\times \; {}_3F_2^T\left(\begin{matrix} B_1-A_0, B_1-A_1, 1 \\ B_1+1/2+s, 1+B_1 \end{matrix} \middle| y\right)$$

using the notation

$${}_pF_q^T\left(\begin{matrix} a_p \\ b_q \end{matrix} \middle| y\right) \equiv \sum_{\ell=0}^{I} \frac{(a_p)_\ell \, y^\ell}{(b_q)_\ell \, \ell!} \tag{5.20a}$$

with

$$I = \mathrm{Min}\{-a_p', \; -b_q'\} \tag{5.20b}$$

where the prime (') indicates that only the members of the set $\{a_p, b_q\}$ that are negative integers are included. For example, the hypergeometric function in (5.19) terminates at the smallest positive value of $\{A_0-B_1, A_1-B_1, -1-B_1\}$. This caveat is required because a hypergeometric function with negative (a_p) and (b_q) parameters does not always terminate. The non-terminating part represents the contribution from the non-overlap region of Figure 1 which has been treated separately. Fortuitously, this series disappears for these cases.

The remaining cases (5.18d-e) are slightly more complicated since they contain a singular part. However, the residues may be evaluated as usual, and the final result is

$$X(M,s-2N) = \frac{-\pi^{\omega}(p^2)^{A_1}(n^2)^{A_2}(p\cdot n)^s\Gamma(1/2+A_2+s)\Gamma(1-B_1+A_1)}{\Gamma(B_1-A_0)\Gamma(-A_2)}$$

$$\Big\{\sum_{\ell=0}^{A_1}\frac{(-y)^{\ell}\Gamma(\ell-A_0)}{\Gamma(\ell+1)\Gamma(1/2+s+\ell)\Gamma(1+A_1-\ell)}\Big[\frac{1}{\varepsilon}+\ln p^2 y - 2\psi(B_1-A_0)$$

$$+ 2\psi(\ell-A_0)-\psi(\ell+1)+\psi(\ell-A_2)-\psi(1/2+s+\ell)\Big]$$

$$+\frac{\Gamma(B_1-A_0)\Gamma(-B_1)\,\theta(-B_1-1)y^{B_1}}{\Gamma(1-B_1+A_1)\Gamma(B_1+1/2+s)}\;{}_3F_2^T\left(\begin{matrix}B_1-A_0,B_1-A_1,1\\ B_1+1/2+s,B_1+1\end{matrix}\Big|\,y\right)\Big\} \qquad (5.21)$$

where the θ-function (cf. (4.7e)) has been inserted into the second term, allowing case (5.18e) to be included in the same expression. The factors multiplying $1/\varepsilon$ reproduce results previously obtained[13]. The complete expression for $S(p,n;-1,0,-1,1)$ was previously given by van Neerven[38]; apart from differences due to normalization (cf. 4.28), (5.21) reproduces his results (his I_{011}, eq. (A.8)).

The case $X(-1,j)$ is of interest ($j \geq 1$). Now we have

$$M = -1$$

instead of (5.17a); (5.17c) carries through with this change. There are only three cases:

$$B_1 = 0,\ A_0 = A_1 = A_2 = -1 \tag{5.22a}$$

$$A_0 \geq 0 > B_1 > A_1 = A_2 \tag{5.22b}$$

$$A_0 \geq 0 = B_1 > A_1 = A_2 \tag{5.22c}$$

The first possibility gives the regular integral X(-1,1) and was treated in Section 5.1. The latter two cases may be obtained from Table 1 (case A.3) by straightforward substitution and will not be given explicitly here.

Finally we come to the integrals Y(i) defined[13)] by

$$Y(i) \equiv S(p,n;-1,-1,N-s,s) = \int d^{2\omega}q(q^2)^{-1}\left((p-q)^2\right)^{-1}(q\cdot n)^{2N-s} \tag{5.23a}$$

with $i = 2N-s$ and

$$N > s-1, \tag{5.23b}$$

giving

$$\begin{aligned} A_0 &= -1-N < 0\ , \\ A_1 &= A_2 = N-s \geq 0\ , \\ B_1 &= 1+N-s > 0\ . \end{aligned} \tag{5.23c}$$

The only possibility for the overlap region is

$$B_1 > A_1 = A_2 > 0 > A_0 \tag{5.23d}$$

and taking the residues for this case we eventually obtain

$$Y(2N-s) = \frac{(-)^{1+A_1}\pi^{\omega}(p^2n^2)^{A_1}(p\cdot n)^s\Gamma(1/2+A_1+s)\Gamma(1+A_1)}{\Gamma(B_1-A_0)}$$

$$\times \sum_{\ell=0}^{A_1} \frac{\Gamma(\ell-A_0)(-y)^{\ell}}{\Gamma(1/2+s+\ell)\Gamma(1+A_1-\ell)\Gamma(\ell+1)} \left[\frac{1}{\varepsilon} + (\ln p^2 - 2\psi(B_1-A_0) + \psi(\ell-A_0))\right] \quad (5.24)$$

The infinite $(O(1/\varepsilon))$ part of this expression reproduces Capper and Leibbrandt's[13] result.

5.4 The General Case

First of all, we note the generalizations of (5.10):

$$\begin{aligned} &S(p,n;K,\mu,\nu,s) = 0, \qquad K \geq 0, \\ &S(p,n;\kappa,M,N,s) = 0, \qquad M \text{ and } N \geq 0, \end{aligned} \quad (5.25)$$

which are results not obtainable in dimensional regularization[13].

Because of the limited number of possible permutations in the overlap region (see Sec. 2.3), it is feasible to classify any S-integral according to the arrangement of the poles and zeros. We expand to first order in ε, and introduce an encompassing notation. Write

$$S(p,n) = \frac{T}{D}\,\mathcal{G} \quad (5.26a)$$

where $\mathcal{G}$ is given in Table 1,

$$T = \pi^{\omega}(p^2)^{2+M+K+N}(n^2)^N(p\cdot n)^s\Gamma(1/2+N+s)\left[1+\varepsilon(\ln p^2-2\overline{\psi}(B_1-A_0)-0\overline{\psi}(B_1-A_1))\right] \quad (5.26b)$$

$$D = \overline{\Gamma}(B_1-A_0)\overline{\Gamma}(B_1-A_1)\overline{\Gamma}(-A_2)\overline{\Gamma}(-A_0-A_1-s), \quad (5.26c)$$

and

$$\overline{\Gamma}(x) = \Gamma(x)\ , \qquad \text{if} \quad x > 0\ ,$$

$$= (-)^x/\Gamma(1-x), \quad \text{if} \quad x \leq 0\ , \qquad (5.26d)$$

$$\overline{\psi}(x) = \psi(x)\ , \qquad \text{if} \quad x > 0\ ,$$

$$= \psi(1-x)\ , \qquad \text{if} \quad x \leq 0\ . \qquad (5.26e)$$

The parameters A_0, A_1, A_2, and B_1 are given in (2.5), and the "0" and "2" coefficients are discussed in the footnote to Table 1. The function $g^{(i)}(b|a)$ appearing in Table 1 is defined by

$$g^{(i)}(b|a) = \sum_{\ell=0}^{b-a} g(a,\ell)\Psi_i(\ell+a)y^\ell \qquad (5.27a)$$

where

$$g(a,\ell) = (-y)^a \frac{\overline{\Gamma}(B_1-a)\prod_{i=0}^{2}\overline{\Gamma}(a-A_i)(a-A_i)_\ell}{\overline{\Gamma}(a+1)\Gamma(a+1/2+s)(a+1/2+s)_\ell(a-B_1+1)_\ell(a+1)_\ell} \qquad (5.27b)$$

with

$$\Psi_0(\ell) = 1\ , \qquad (5.27c)$$

$$\Psi_1(\ell) = 1 + \varepsilon\ \overline{\psi}(B_1-\ell) + \varepsilon_0\ \overline{\psi}(-A_0+\ell) + \varepsilon_1\ \overline{\psi}(-A_1+\ell)\ , \qquad (5.27d)$$

$$\Psi_2(\ell) = -1-\varepsilon\Big(\ell n\ y - \overline{\psi}(-\ell) + \overline{\psi}(-A_2+\ell) - \psi(\tfrac{1}{2}+s+\ell) + 2\overline{\psi}(-A_0+\ell) + 0\overline{\psi}(-A_1+\ell)\Big), \qquad (5.27e)$$

$$\Psi_3(\ell) = \big(\Psi_1(\ell)+\Psi_2(\ell)\big)/\varepsilon$$

$$= \overline{\psi}(B_1-\ell)+\overline{\psi}(-\ell)-\overline{\psi}(-A_0+\ell)-\overline{\psi}(-A_1+\ell)-\overline{\psi}(-A_2+\ell)+\psi(\tfrac{1}{2}+s+\ell)-\ell n\ y\ , \qquad (5.27f)$$

and the convention that $g^i(b|a) = 0$ if $b < a$. The function Z in Table 1 is defined by

$$Z(A_0,A_1,B_1-1) = \sum_{\ell} g(a,\ell)\Psi_3(\ell+a)y^{\ell} , \qquad \text{if } |y| \leq 1 ,$$

$$= \sum_{\ell} \tilde{g}(\ell+d)\Psi_4(\ell+d)(-y)^{-\ell-d}, \quad \text{if } |y| > 1 , \tag{5.28a}$$

where

$$a = \mathrm{Max}(0,B_1,A_0+1,A_1+1) , \tag{5.28b}$$

$$d = \mathrm{Max}(0,1-B_1+A_2,1+A_2) , \tag{5.28c}$$

with

$$\tilde{g}(\ell) = (\tfrac{1}{2})(-)^{1-A_0-A_1}$$

$$\times \frac{\Gamma(B_1-A_2+\ell)\Gamma(-A_2+\ell)}{\Gamma(1+\ell)\Gamma(1+A_1-A_2+\ell)\Gamma(1+A_0-A_2+\ell)\Gamma(1/2+A_2+s-\ell)} \tag{5.28d}$$

and

$$\Psi_4(\ell) = \Big[\psi(B_1-A_2+\ell)+\psi(\ell-A_2)-\psi(1+\ell)-\psi(1+A_1-A_2+\ell)$$

$$-\psi(1+A_0-A_2+\ell)+\psi(\tfrac{1}{2}+A_2+s-\ell) + \ln\frac{1}{y}\Big]^2$$

$$+ \pi^2+\psi'(B_1-A_2+\ell)+\psi'(\ell-A_2)-\psi'(1+\ell)$$

$$- \psi'(1+A_1-A_2+\ell)-\psi'(1+A_0-A_2+\ell)-\psi'(\tfrac{1}{2}+A_2+s-\ell) . \tag{5.28e}$$

Nonzero values for the functions $\mathcal{G}$ in (5.26a) are given in Table 1.

Although it is possible to extend Table 1 to higher orders in ρ, τ and σ in order to calculate exponent derivatives, we refrain from doing so because the general expressions rapidly become unmanageably lengthy and complicated. However the general evaluation may be executed by computer[54] (SCHOONSCHIP[40]). We give a few examples of first order exponent derivatives in the following subsections.

5.5 Exponent Derivatives in Covariant Gauge

To illustrate, we consider three distinct cases of first order exponent derivatives.

(a) $\kappa = -1$, $\mu = 1$. As implied by (5.10b), S is of $O(\sigma/\varepsilon_1)$, vanishing in the limit $\sigma = 0$. The μ-derivative is non-zero, however. We find

$$S(p;-1,1) = \int d^{2\omega}q \, \frac{q^2}{(p-q)^2} = 0, \tag{5.29}$$

$$\frac{d}{d\kappa} S(p;\kappa,1)\Big|_{\kappa=-1} = \int d^{2\omega}q \, \frac{q^2 \ln(p-q)^2}{(p-q)^2} = 0, \tag{5.30}$$

$$\frac{d}{d\mu} S(p;-1,\mu)\Big|_{\mu=1} = \int d^{2\omega}q \, \frac{q^2 \ln q^2}{(p-q)^2} = -\frac{\pi^\omega (p^2)^2}{6}\left[\frac{1}{\varepsilon} + \gamma + \ln p^2 - \frac{11}{3}\right] \tag{5.31}$$

Note that the non-vanishing exponent derivative is of $O(1/\varepsilon)$, rather than $O(1/\varepsilon^2)$, as is usually the case.

(b) $\kappa = \mu = -1$. S is symmetric under interchange $\kappa \leftrightarrow \mu$, and has an $O(1/\varepsilon_1)$ singularity. We find

$$S(p;-1,-1) = \int d^{2\omega}q\,[q^2(p-q)^2]^{-1} = -\pi^{\omega}[\tfrac{1}{\varepsilon} + \gamma + \ln p^2 - 2], \tag{5.32}$$

$$\frac{d}{d\kappa} S(p;\kappa,-1)\Big|_{\kappa=-1} = \int d^{2\omega}q\, \frac{\ln(p-q)^2}{(p-q)^2 q^2}$$

$$= \frac{d}{d\mu} S(p;-1,\mu)\Big|_{\mu=-1} = \int d^{2\omega}q\, \frac{\ln q^2}{(p-q)^2 q^2}$$

$$= \pi^{\omega}[\frac{1}{\varepsilon^2} + \frac{\gamma-1}{\varepsilon} + \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{1}{2}\ln^2 p^2 - \gamma + \ln p^2]. \tag{5.33}$$

(c) $\kappa = -1$, $\mu = -2$. S has an $O(1/\varepsilon_0)$ singularity. We follow the prescription described in Sec. 4.3 and convert it to a $1/\varepsilon_1$ singularity before differentiation to rid the exponent derivatives of logarithmic infinite terms. The results are

$$S(p;-1,-2) = \int d^{2\omega}q\,[(p-q)^2 q^4]^{-1} = \frac{\pi^{\omega}}{p^2}\,(\frac{1}{\varepsilon} + \gamma + \ln p^2), \tag{5.34}$$

$$\frac{d}{d\kappa} S(p,\kappa,-2)\Big|_{\kappa=-1} = \int d^{2\omega}q\, \frac{\ln(p-q)^2}{(p-q)^2 q^4}$$

$$= -\frac{\pi^{\omega}}{p^2}\,[\frac{1}{\varepsilon^2} + \frac{\gamma}{\varepsilon} + \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{1}{2}\ln^2 p^2], \tag{5.35}$$

$$\frac{d}{d\mu} S(p,-1,\mu)\Big|_{\mu=-2} = \int d^{2\omega}q \, \frac{\ell n q^2}{(p-q)^2 q^4}$$

$$= -\frac{\pi^\omega}{p^2}\left[\frac{1}{\varepsilon^2} + \frac{\gamma-1}{\varepsilon} + \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{1}{2}\,\ell n^2 p^2 - \gamma - \ell n p^2\right].$$

(5.36)

5.6 Exponent Derivatives in Axial Gauge

Consider the case $\kappa = \mu = \nu = -1$, $s = 0$. The integral belongs to Class A, case A.4 of Table 1 with $(A_0 \leftrightarrow A_1)$ and has an $O(1/\varepsilon_0)$ singularity. We find, after changing $1/\varepsilon_0 \to -1/\varepsilon_1$,

$$S(-1,-1,-1,0) = \int d^{2\omega}q\,[(p-q)^2 q^2 (\ell\cdot n)^2]^{-1}$$

$$= -\frac{2\pi^\omega}{p^2 n^2}\left[\frac{1}{\varepsilon} + \gamma + \ell n(p^2/4y)\right], \qquad (5.37)$$

$$\frac{d}{d\kappa} S(\kappa,-1,-1,0)\Big|_{\kappa=-1} = \int d^{2\omega}q\,[(p-q)^2 q^2 (q\cdot n)^2]^{-1}\,\ell n(p-q)^2$$

$$= \frac{2\pi^\omega}{p^2 n^2}\left[\frac{1}{\varepsilon^2} + \frac{\gamma}{\varepsilon} + \frac{\gamma^2}{2} - \frac{\pi^2}{4} - \frac{1}{2}\,\ell n^2(p^2/4y) + \ell n^2 4y + Z_1\right] \qquad (5.38)$$

$$\frac{d}{d\mu} S(-1,\mu,-1,0)\Big|_{\mu=-1} = \int d^{2\omega}q\,[(p-q)^2 q^2 (q\cdot n)^2]^{-1}\,\ell n q^2$$

$$= \frac{2\pi^\omega}{p^2 n^2}\left[\frac{1}{\varepsilon^2} + \frac{\gamma}{\varepsilon} + \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{1}{2}\,\ell n^2(p^2/4y) + \frac{1}{2}\,\ell n^2 4y + Z_1\right] \qquad (5.39)$$

$$\left(\frac{d}{d\nu} - \frac{d}{d\mu} - \ln n^2\right) S(-1,\mu,\nu,0)\Big|_{\substack{\mu=-1\\ \nu=-1}} = \int d^{2\omega}q\,[(p-q)^2 q^2 (q\cdot n)^2]^{-1} \ln[(q\cdot n)^2/q^2n^2]$$

$$= \frac{2\pi^\omega}{p^2n^2}\left\{(\ln 4-2)\left[\frac{1}{\varepsilon} + \gamma + \ln(p^2/4y)\right] + \frac{1}{2}\ln^2 4y + Z_1\right\} \tag{5.40}$$

where as always $y \equiv (p\cdot n)^2/p^2n^2$ and

$$Z_1 \equiv \sqrt{\pi} \sum_{\ell=1}^{\infty} \frac{y^\ell \Gamma(\ell)}{\Gamma(\frac{1}{2}+\ell)} \left[\psi(\ell) - \psi(\tfrac{1}{2}+\ell) + \ln y\right], \quad |y| < 1,$$

$$\equiv \frac{-1}{(2\sqrt{\pi})} \sum_{\ell=1}^{\infty} \frac{y^{-\ell}\Gamma(1/2+\ell)}{\Gamma(1+\ell)} \left\{[\psi(1+\ell) - \psi(\tfrac{1}{2}-\ell) + \ln y]^2 + \pi^2 - \psi'(2+\ell) - \psi'(-\tfrac{1}{2}-\ell)\right\}, \quad |y|>1. \tag{5.41}$$

The particular logarithm in the integrand in (5.40) is chosen because logarithmic dependence on the gauge vector n occurs only via the expression $\ln y$. It is worthwhile pointing out that the result (5.40) contains neither poles of $O(1/\varepsilon^2)$ nor logarithms of $O(\ln^2 p^2)$, each of which is a signature of overlapping ultraviolet divergencies. This is another consequence of our regularization method in which physical (ultraviolet) and unphysical (axial gauge) singularities are completed decoupled. Note that a single pole always appears in the combination $\varepsilon^{-1} + \gamma + \ln p^2$, and a double pole, $\varepsilon^{-2} + \gamma\varepsilon^{-1} + \gamma^2/2 - \pi^2/12 - (\ln^2 p^2)/2$. Note further that here the magnitude of the ratio of the coefficient of the $\ln^2 p^2$ term to that of the ε^{-2} term is a factor of two less than the ratio given by the 'tHooft-Veltman prescription (cf. Appendix D).

6. BEYOND POLYLOGS

We have seen that primal S integrals and their exponent derivatives may be useful in nonperturbation theories because they generate the same set of polylogs as are generated by loop-integrals in perturbation theories. On the other hand, it is altogether probable that integro-differential equations derived from nonperturbation theories admit solutions with nonintegral exponents. There is therefore a need to study such S-integrals (with nonintegral exponents) in their own right, rather than only as vehicles for generating polylogs. Furthermore, it is obvious that such integrals generate functions that are qualitatively different from those generated by primal S-integrals; this possibility suggests some interesting conjectures.

A particularly significant property of S-integrals is that they are free from pole singularities when none of the three indices α_0, α_1 and $\alpha_3 \equiv \alpha_2-\beta_1$ (Sec. (2.5)) is a non-negative integer (in this section the limit $\omega \to 2$ is always implied). Does this imply that if integro-differential equations derived from a nonperturbation theory admit solutions with nonintegral exponents, then the theory may be divergentless or finite, eliminating the need for renormalization? A less radical possibility[41)] is that the α-indices remain as integers, but their components κ, μ, ν, $\cdots$ are not. We consider the second possibility first.

Suppose the solution has a factor $(p^2)^\mu$ with the nonintegral exponent $\mu = M+\sigma$. Then upon expansion in σ,

$$(p^2)^\mu = (p^2)^M\left(1 + \sigma \ell n p^2 + \frac{\sigma^2}{2} \ell n^2 p^2 + \cdots\right). \qquad (6.1)$$

The right-hand-side has the appearance of a polylog, with σ playing the role of a coupling constant in a renormalized perturbation theory. This suggests the possibility that loop-expansions in perturbation series may be mere approximations to an expansion in σ. Could the dimensionless σ then be a "fixed point"[42] of the theory?

With regards to the possibility of constructing a finite theory, consider now the ultraviolet index α_1, which governs the only type of persistent and physical singularity in S-integrals, or as discussed in Sec. 4.3, in any Feynman integral. The index represents the overall dimensionality of the integral to which it belongs. In perturbation theory Feynman integrals are representations of physical amplitudes which always have integral dimensions. Therefore α_1 is always an integer in perturbation theories. Consider an integral $S(\alpha_1)$ representing a physical amplitude $Q(A_1)$ of integral dimension A_1. Then in a perturbation theory the relation between Q and S may be expressed, without loss of generality, as

$$Q(A_1) = S_{ren}(\alpha_1)\Big|_{\alpha_1=A_1}, \tag{6.2}$$

where the subscript "ren" means the integral is renormalized when $A_1 \geq 0$. Suppose the corresponding nonperturbation theory admits a solution such that

$$\alpha_1 = A_1 - \varepsilon_1$$

where ε_1 is not an integer. Then first of all S is finite and no renormalization is needed. The continuous exponent ε_1 may again be a fixed point of the theory. However since the dimensionality of Q cannot change (6.2) must now read

$$Q(A_1) = \Lambda^{\varepsilon_1} S(A_1 - \varepsilon_1), \tag{6.3}$$

where Λ has the dimension of momentum. It follows that a finite theory implies the existence of a dimensional scale parameter Λ. The symbiotic relation between renormalization and a scale parameter is well known[43]. In perturbative QCD[6], the parameter is referred to as the momentum of subtraction, and is not a theoretically calculable quantity there.

Finally, because of the richness and compactness of the G-function notation, it may prove that a similarly compact result can be found to describe entire physical amplitudes by analytically summing a series of G-functions.

7. SUMMARY

A hybrid of analytic and dimensional regularization has been used to discover a compact G-function representation ((2.2)) for a large class of divergent Feynman integrals, or S-integrals ((1.1)), which occur in the calculation of two-point functions in massless gauge theories in covariant and axial gauges; integrals in the light-cone gauge appear as a special limit of the analytic continuation of the axial gauge. For integrals in the axial gauge, our method of analytically regulating the spurious singularity is particularly useful; it is far simpler than the old method of evaluating the integrals by the principal value prescription, which is shown to correspond to operating a regulator - in this case a polynomial in a differential operator - on a sum of G-functions. In addition, our method permits the evaluation of Feynman integrals with logarithmic factors in the integrand. The infinite and finite parts of all S-integrals are given in (5.26-28) and Table 1.

It is pointed out that the set of polylogs - power series in $(p^2)^m \ell n^j p^2$ - generated in multi-loop Feynman integrals in perturbation theory coincides with the set of exponent derivatives of one-loop Feynman integrals: polylogs from N-loop integrals are equivalent to those generated by the $(N-1)^{th}$ order exponent derivatives of one-loop integrals. Since the G-function representation for the S-integrals is an analytic function of all the exponents, the computation of exponent derivatives is in principle straightforward. Some examples of first order derivatives are given. The significance of the possibility of generating polylogs independently of perturbation expansions was pointed out; the polylogs may be used as a basis for the solutions of integro-differential equations of nonperturbation theories.

A striking similarity between the 'tHooft-Veltman prescription for dealing with overlapping divergencies and the exponent derivative is pointed out and explored. A proof asserting that their prescription indeed eliminates all logarithmic infinite parts in all multi-loop integrals is given (Appendix D). It is also shown that, when the S-integrals are judiciously regulated, the exponent derivatives contain high order but no logarithmic infinite parts. Furthermore, all high order singularities arise solely from ultraviolet divergencies. The absence of logarithmic infinite parts implies that nonperturbation theories, with physical amplitudes expressed in terms of S-integrals and their exponent derivatives, are renormalizable - all infinite parts may be cancelled by counterterms.

Although a hybrid regularization proved to be the most convenient method of regularization for the task on hand, all that has been achieved could have been accomplished by analytic regularization alone, but not by dimensional regularization. The latter is deficient for the present task on two counts: it does not allow the analytic regularization of the spurious singularities of the axial gauge thus leading to unnecessarily lengthy and treacherous computation, and it does not permit the computation of exponent derivatives which, as pointed out earlier, may be of profound significance in nonperturbation theories. On the other hand, for S-integrals without derivatives, it is observed that analytic regularization and dimensional regularization lead to identical results, provided that in the latter the Dirac algebra is done in an integral - i.e. not 2ω - dimensional space, and the factor π^{ω} that appears in the integral is restored to π^2. Since the replacement of dimensional regularization by analytic regularization is a simple matter of choice, this observation eliminates the need for ever doing the Dirac algebra in nonintegral dimensional space, thereby solving the contentious problem in dimensional regularization arising from attempting to consistently define antisymmetric tensors in a continuous dimensional space and leading to spurious γ_5-anomalies. The equivalence of the two methods also implies that analytic regularization preserves gauge invariance.

By considering integrals with nonintegral exponents, we are led to some intriguing observations. Notable among these is the possibility of a finite nonperturbation theory emerging naturally.

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TABLE 1[§] - Evaluation of $\mathcal{G}$ in Eq. (5.26a)

Case	Condition	Infinite Part[††]	Regular Part
Class A[†]: $A_2 < 0$			
A.1	$B_1=0>A_0\geq A_1$	-	$Z(A_0,A_1,B_1-1)$
A.2	$A_0\geq B_1>0>A_1$	$\frac{1}{\varepsilon_0}\, g^{(1)}(A_0\|B_1)$	$2g^{(0)}(B_1-1\|0) + g^{(3)}(A_0\|B_1)$
A.3	$A_0\geq 0\geq B_1>A_1$	$\frac{1}{\varepsilon_0}\, g^{(1)}(A_0\|0)$	$-g^{(0)}(-1\|B_1) + g^{(3)}(A_0\|0)$
A.4	$A_0\geq 0>A_1\geq B_1$	-	$-g^{(0)}(A_1\|B_1) - 0g^{(0)}(A_0\|0)$
A.5	$0>A_0\geq A_1\geq B_1$	-	$-g^{(0)}(A_1\|B_1)$
A.6	$A_0\geq 0>A_1>A_2\geq B_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_2\|B_1)$	$g^{(0)}(A_1\|A_2+1)-0g^{(0)}(A_0\|0)$
A.7	$A_0\geq 0>A_2\geq B_1>A_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_2\|B_1)$ $+ \frac{1}{\varepsilon_0}\, g^{(1)}(A_0\|0)$	$-g^{(0)}(-1\|A_2+1)+g^{(3)}(A_0\|0)$
A.8	$A_0\geq 0>A_2\geq A_1\geq B_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_1\|B_1)$	$-0g^{(0)}(A_2\|A_1+1)-0g^{(0)}(A_0\|0)$

TABLE 1 - continued

A.9	$0>A_0\geq A_1\geq A_2\geq B_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_2\|B_1)$	$-g^{(0)}(A_1\|A_2+1)$
A.10	$0>A_0\geq A_2>A_1\geq B_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_1\|B_1)$	$0g^{(0)}(A_2\|A_1+1)$
A.11	$0>A_2\geq A_0\geq B_1>A_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_0\|B_1)$	$-g^{(0)}(-1\|A_2+1)-2g^{(0)}(A_2\|A_0+1)$
A.12	$0>A_2>A_0>A_1\geq B_1$	$\frac{1}{\varepsilon_3}\, g^{(2)}(A_1\|B_1)$	$-0g^{(0)}(A_0\|A_1+1)$

Class $B^{\dagger}$: $A_2 \geq 0$

B.1	$A_2\geq B_1\geq 0>A_0\geq A_1$	$\frac{1}{\varepsilon_3} g^{(1)}(A_2\|B_1)$	$g^{(0)}(B_1-1\|0)$
B.2	$A_2\geq B_1>A_0\geq 0>A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0\|0) + \frac{1}{\varepsilon_3}\, g^{(1)}(A_2\|B_1)$	$g^{(0)}(B_1-1\|A_0+1)$
B.3	$A_2>A_0\geq B_1\geq 0>A_1$	$(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}) g^{(1)}(A_0\|B_1)$	$2g^{(0)}(B_1-1\|0)+2g^{(0)}(A_2\|A_0+1)$
B.4	$A_2>A_0\geq 0>B_1>A_1$	$(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}) g^{(1)}(A_0\|0)$	$2g^{(0)}(A_2\|A_0+1)$
B.5	$A_2\geq 0>B_1>A_0\geq A_1$	$\frac{1}{\varepsilon_3} g^{(1)}(A_2\|0)$	-

TABLE 1 - continued

B.6	$A_2 \geq 0 > A_0 \geq B_1 > A_1$	-	$2g^{(0)}(A_2 \| 0)$
B.7	$B_1 > A_2 \geq 0 > A_0 \geq A_1$	-	$g^{(0)}(A_2 \| 0)$
B.8	$B_1 > A_2 > A_0 \geq 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_0 \| 0)$	$g^{(0)}(A_2 \| A_0{+}1)$
B.9	$B_1 > A_0 \geq A_2 \geq 0 > A_1$	$\frac{1}{\varepsilon_0} g^{(1)}(A_2 \| 0)$	-
B.10	$A_0 \geq A_2 \geq B_1 \geq 0 > A_1$	$(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}) g^{(1)}(A_2 \| B_1)$	$2g^{(0)}(B_1{-}1 \| 0)$
B.11	$A_0 \geq A_2 \geq 0 > B_1 > A_1$	$(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon_3}) g^{(1)}(A_2 \| 0)$	-
B.12	$A_0 \geq B_1 > A_2 \geq 0 > A_1$	-	$2g^{(0)}(A_2 \| 0)$

§ Symbols and notation: for A_i, B_1 and ε_i see (2.5); $\varepsilon_3 \equiv \varepsilon_b + \varepsilon_2$; for $g^{(i)}(a|b)$ see (5.27), for Z see (5.28). In the limit $\rho = \sigma = \tau = 0$, $\varepsilon_0 = \varepsilon_3 = \varepsilon$ and $\varepsilon_1 = -\varepsilon$. $\mathcal{G}$ is the sum of infinite and regular parts.

† Class A cases all have $A_2 < 0$, $A_0 > A_1$ and $B_1 > A_2$ if the position of A_2 is not given; if $A_0 + A_1 + s \geq 0$, then $\mathcal{G} = 0$. Class B cases all have $A_2 \geq 0$, $A_0 \geq A_1$ and $B_1 > A_1$; if $A_1 \geq B_1$ then $\mathcal{G} = 0$. Corresponding expressions for $A_1 > A_0$ are obtained by interchanging A_0 and A_1 and the coefficients 2 and 0, and replacing ε_0 by ε_1, in the table and in (5.26) and (5.27).

†† The "infinite" part given here includes terms of $O(1/\varepsilon)$ as well as those $O(1)$ terms that are naturally associated with the $O(1/\varepsilon)$ terms.

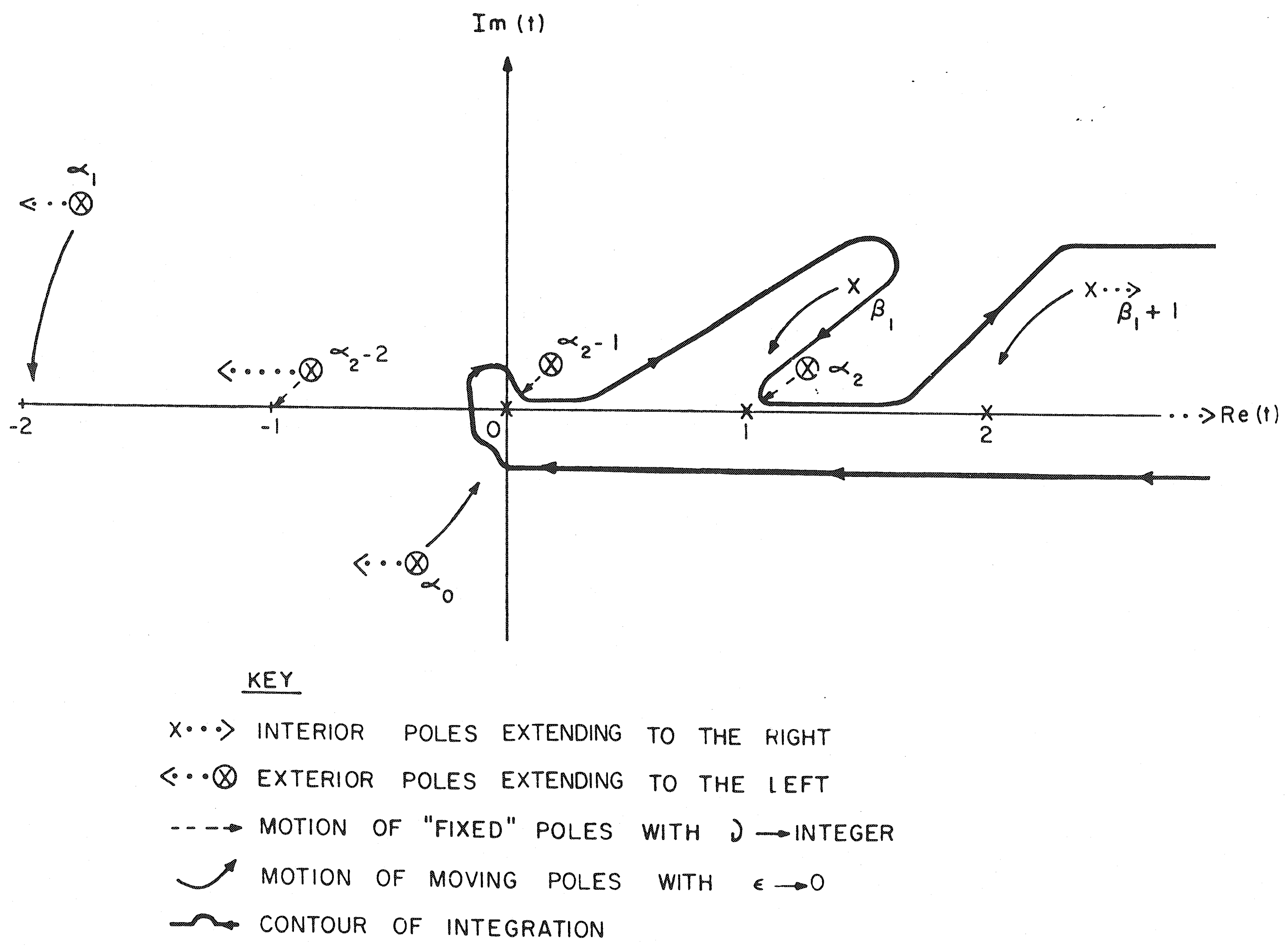

Figure 1. Pole structure of the contour integral in (2.8) for the case $A_0 = 0$, $A_1 = -2$, $A_2 = 1$ and $B_1 = 1$, corresponding to $S(p,n; -2,-3,1,0)$ with $|y| < 1$.

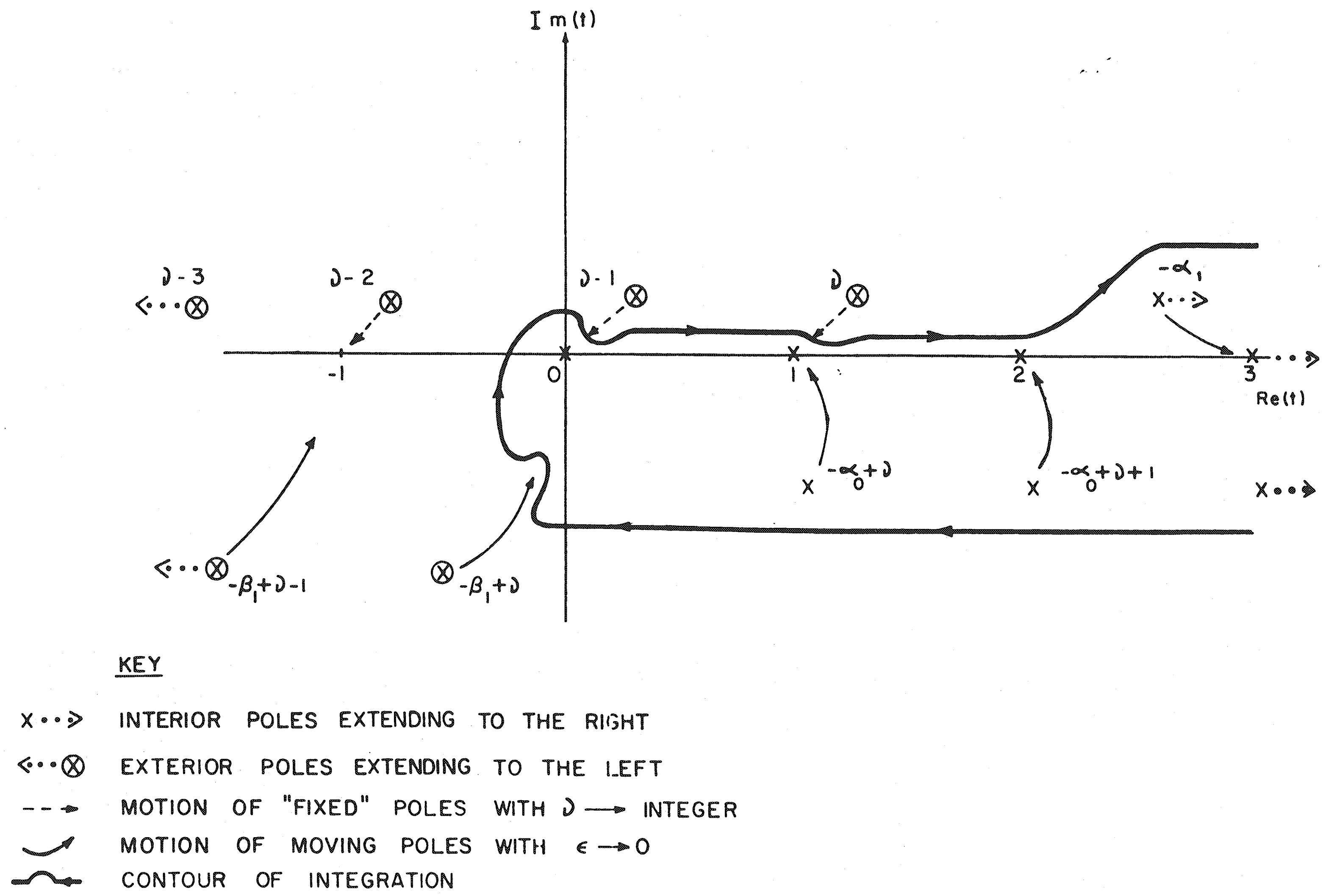

Figure 2. Pole structure of the contour integral in (2.12) for the case $A_0 = 0$, $A_1 = 2$, $A_2 = 1$, $B_1 = 1$ corresponding to $S(p,n; -2,-3,1,0)$ with $|y| > 1$.

APPENDIX A

In this and Appendix B, the main result (2.2) will be derived. We begin by replacing three of the factors in (2.1) by an integral representation of the form

$$(q^2)^\mu = \frac{1}{\Gamma(-\mu)} \int_0^\infty t^{-\mu-1}\, e^{-q^2 t}\, dt, \qquad \mathrm{Re}(\mu) < 0 \tag{A.1}$$

and transpose with the outermost integral to obtain

$$S_{2\omega}(p,n;\kappa,\nu,\mu,s) = \frac{1}{\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\kappa)} \int_0^\infty dt \int_0^\infty du \int_0^\infty dv\; t^{-\mu-1} u^{-\nu-1} v^{-\kappa-1} e^{-p^2 v} J(t,u,v)$$

$$\mathrm{Re}(\mu,\nu,\kappa) < 0, \tag{A.2}$$

where

$$J(t,u,v) = \int d^{2\omega}q (q\cdot n)^s \exp(-q^2(t+v) + 2(p\cdot q)v - (q\cdot n)^2 u). \tag{A.3}$$

The above integral has been derived elsewhere[13]; it is equivalent to

$$\int d^{2\omega}q\; (q\cdot n)^s\; e^{-\alpha q^2 + 2\beta p\cdot q - \gamma (q\cdot n)^2}$$

$$= \left(\frac{\pi}{\alpha}\right)^\omega \left(\frac{\beta p\cdot n}{\alpha+\gamma n^2}\right)^s \frac{\alpha^{1/2}}{(\alpha+\gamma n^2)^{1/2}} \exp\left(\left[\beta^2 p^2 - \frac{\gamma\beta^2 (p\cdot n)^2}{\alpha+\gamma n^2}\right]/\alpha\right) \tag{A.4}$$

which has the feature that it reproduces the usual results whenever ω is a positive half-integer. In point of fact, we may always take ω equal to a positive half-integer in (A.3) and (A.4), provided μ, ν and κ are continuous variables, as follows from the discussion in Sect. 2. Thus (A.2) becomes

$$S_{2\omega}(p,n;\kappa,\nu,\mu,s) = \frac{\pi^{\omega}\,(p\cdot n)^{s}}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)} \int_0^\infty dt \int_0^\infty du \int_0^\infty dv\, \frac{t^{-\mu-1}u^{-\nu-1}v^{-\kappa-1+s}(t+v)^{1/2-\omega}}{(t+v+un^2)^{1/2+s}}$$

$$\times \exp\left[\frac{p^2v^2}{t+v} - \frac{uv^2(p\cdot n)^2}{(t+v)(t+v+un^2)} - p^2v\right] \qquad (A.5)$$

Now transform the variables according to

$$\begin{aligned} t &= (1-\tau)\,\xi\lambda\ , \\ u &= \lambda\tau/n^2, \\ v &= \lambda(1-\tau)(1-\xi), \end{aligned} \qquad (A.6)$$

and the scale integration (λ, from 0 to ∞) may be evaluated analytically using (A.1). The eventual result is

$$S_{2\omega}(p,n;\kappa,\nu,\mu,s) = \frac{\pi^{\omega}(n^2)^{\nu}(p\cdot n)^{s}(p^2)^{\omega+\mu+\kappa+\nu}\Gamma(-\omega-\mu-\nu-\kappa)}{\Gamma(-\mu)\,\Gamma(-\nu)\,\Gamma(-\kappa)}$$

$$\times \int_0^1 d\tau \int_0^1 d\xi\ \tau^{-\nu-1}(1-\tau)^{-1/2+\nu+s}\,\xi^{-\mu-1}(1-\xi)^{\omega+\mu+\nu+s-1}$$

$$\times\ [\xi + \tau y(1-\xi)]^{\omega+\mu+\nu+\kappa}\ , \qquad (A.7)$$

with

$$y = (p\cdot n)^2/(p^2n^2),$$

if

$$\mathrm{Re}(\omega+\mu+\nu+\kappa) < 0 \ . \tag{A.8}$$

The double integral in (A.7) is in the canonical form of an integral evaluated in Appendix B, with the substitutions

$$\begin{aligned}
\alpha &\to -(\omega+\kappa+\mu+2\nu+1) \\
\beta &\to -1/2+\nu+s, \\
\gamma &\to \omega+\mu+\nu+s-1 \\
\mu &\to \mu+1, \\
\zeta &\to \omega+\mu+\nu+\kappa \ .
\end{aligned} \tag{A.9}$$

In contrast to our approach, Bollini et al.[14] uses a generalized Feynman formula instead of the generalized exponentiation of (A.1). Like us, they use analytic continuation to define divergent integrals. Speer[14] regulates propagators (as opposed to only integrals), uses (A.1) in a modified form - the lower limit of integration is replaced by r and the limit $r \to 0^+$ is considered - and does not explicitly use the principle of analytic continuation. For the implication of this subtle difference between Speer's and our approach see Sect. 4.4.

APPENDIX B

We wish to evaluate the double integral

$$\mathcal{J} = \int_0^1 d\xi \int_0^1 d\tau\ \tau^{\alpha+\zeta}(1-\tau)^{\beta}(1-\xi)^{\gamma}\xi^{-\mu}D_y^{\zeta} \tag{B.1}$$

where

$$D_y = \xi + (1-\xi)\tau y\ . \tag{B.2}$$

Perform the transformation $v = (1-\xi)/\xi$, and recognizing the τ integral as the integral representation of a hypergeometric function[44] obtain

$$\mathcal{J} = \frac{\Gamma(\alpha+\zeta+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+\zeta+2)} \int_0^\infty dv\ v^{\gamma}(1+v)^{\mu-2-\gamma-\zeta}\,{}_2F_1(-\zeta,\alpha+\zeta+1;\alpha+\beta+\zeta+2;-vy)\,. \tag{B.3}$$

Express the hypergeometric function as a G-function[45] so that

$$\mathcal{J} = \frac{\Gamma(\beta+1)}{\Gamma(-\zeta)} \int_0^\infty dv\ v^{\gamma}(1+v)^{\mu-2-\gamma-\zeta} G^{1,2}_{2,2}\left(vy\middle|\begin{matrix}1+\zeta,-\alpha-\zeta;\\ 0;-1-\alpha-\beta-\zeta\end{matrix}\right)\,. \tag{B.4}$$

This integral is known[46], and the final result is

$$\mathcal{J} = \frac{\Gamma(\beta+1)}{\Gamma(-\zeta)\Gamma(2+\gamma+\zeta-\mu)}\, G^{2,3}_{3,3}\left(y\middle|\begin{matrix}-\gamma,1+\zeta,-\alpha-\zeta;\\ 0,1+\zeta-\mu;-1-\alpha-\beta-\zeta\end{matrix}\right), \tag{B.5}$$

if

$$\mathrm{Re}(2+2\zeta+\alpha-\mu) > 0, \tag{B.6a}$$

$$\mathrm{Re}(\gamma) > -1, \tag{B.6b}$$

$$\mathrm{Re}(\mu) < 1, \tag{B.6c}$$

$$\mathrm{Re}(1+\beta) > 0, \tag{B.6d}$$

$$\mathrm{Re}(\alpha+\zeta) > -1 \ . \tag{B.6e}$$

The conditions (B.6a-e) may be relaxed on the right hand side of (B.5), since the G-function is well-defined for all values of its parameters.

APPENDIX C

We wish to show the equivalence of van Neerven's representation[38] for $I_{111} = \frac{i}{16\pi^4} S(p,n;-1,-1,-1,1)$ and our (5.1). According to van Neerven,

$$I_{111} = \frac{i}{16\pi^2} \frac{1}{p\cdot n} F(x) \tag{C.1}$$

where

$$F(x) = 2 \int_0^1 dt \frac{1}{1 + t^2(x-1)} \ell n\left(\frac{1+t}{1-t}\right) , \qquad 1 < x < \infty . \tag{C.2}$$

Identify the logarithmic term with a well-known power series in hypergeometric form[47] and obtain

$$F(x) = 2 \int_0^1 dv \, (1-(1-x)v)^{-1} \, {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; v\right) \tag{C.3}$$

after an obvious transformation of variables. Now impose a linear transformation ($v \to v/(v-1)$) on the hypergeometric function and again transform the variables ($t = v/(1-v)$), obtaining

$$F(x) = 2 \int_0^\infty dt(1+xt)^{-1} \, {}_2F_1(1,1; \tfrac{3}{2};-t) . \tag{C.4}$$

Write the hypergeometric function as a G-function,[45] and use a known integration formula[46]. Thus

$$F(x) = \frac{\Gamma(1/2)}{x} G^{2,3}_{3,3}\left(\frac{1}{x}\left|\begin{matrix}0,0,0;\\0,0;-1/2\end{matrix}\right.\right), \qquad \frac{1}{x} < 1, \tag{C.5}$$

demonstrating that

$$I_{111} = i\,\frac{\Gamma(1/2)p\cdot n}{16\pi^2\; p^2 n^2}\; G^{2,3}_{3,3}\left(y\left|\begin{matrix}0,0,0;\\0,0;-1/2\end{matrix}\right.\right), \qquad y \leq 1, \tag{C.6}$$

using $y = 1/x$. This is precisely (5.1), after allowing for the different normalization. By the principles of analytic continuation, both representations are valid for all values of y.

APPENDIX D

We wish to show that the 'tHooft-Veltman[25] prescription removes all logarithmic infinite parts in any multi-loop integral with overlapping divergencies. We shall use an economical notation where all finite parts and factors with integral exponents are suppressed; only infinite parts and factors of q^2 with the non-integral parts of the exponents are retained. The "leading to" symbol "$\sim$" will be used whenever this notation is in force, thus reserving the equality symbol "=" to have its usual meaning. We shall also use the same symbol for the inner (or integrated) and outer (or external) momenta. Thus a divergent integral[25]

$$\int d^{2\omega}q\,[(p-q)^2]^{K+\rho}\cdots = \frac{(p^2)^{\varepsilon+\rho}}{\varepsilon+\rho}(\cdots) + \cdots \tag{D.1a}$$

where ρ is any continuous small variable, is symbolically expressed as

$$\int (q^2)^{\rho} \sim \frac{(q^2)^{\varepsilon+\rho}}{\varepsilon+\rho} \; . \tag{D.1b}$$

Without analytic regularization, consider a one-loop integral generating an infinite part,

$$\int^{\text{1-loop}} 1 \sim \frac{(q^2)^{\varepsilon}}{\varepsilon} \; . \tag{D.2}$$

The corresponding two-loop integral, together with the 'tHooft-Veltman subtraction and (D.1b), reads

$$\int^{2\text{-loop}} 1 \sim \frac{1}{\varepsilon} \int [(q^2)^{\varepsilon} - 1] \sim \frac{1}{\varepsilon} \left[\frac{(q^2)^{2\varepsilon}}{2\varepsilon} - \frac{(q^2)^{\varepsilon}}{\varepsilon}\right]$$

$$= - \frac{1}{2\varepsilon^2} + \frac{1}{2} \ln^2 q^2 + O(\varepsilon), \tag{D.3}$$

which shows that the subtraction indeed renders the expansion in (D.3) free of the potential logarithmic infinite part $(\ln q^2)/\varepsilon$.

Assume the N-loop integral, with all subtractions, yields

$$\int^{N\text{-loop}} 1 \sim \frac{1}{\varepsilon^N} \sum_{m=1}^{N} a_m^{(N)} (q^2)^{m\varepsilon} . \tag{D.4}$$

Then the subtracted (N+1)-loop integral, with repeated application of (D.1b), is

$$\int^{(N+1)\text{-loop}} 1 \sim \frac{1}{\varepsilon^N} \sum_{m=1} a_m^{(N)} \int [(q^2)^{m\varepsilon} - 1]$$

$$\sim \frac{1}{\varepsilon^{N+1}} \sum_{m=1}^{N} a_m^{(N)} \left[\frac{(q^2)^{(m+1)\varepsilon}}{m+1} - (q^2)^{\varepsilon}\right]. \tag{D.5}$$

We claim that the solution is

$$a_m^{(N)} = \frac{(-)^{N-m}}{m!(N-m)!} , \qquad m = 1, \cdots, N. \tag{D.6}$$

This is easily proven by induction using (D.1b) and the relation[48)]

$$\sum_{m=1}^{N} \frac{(-)^m m^\ell}{m!(N-m)!} = -\frac{\delta_{\ell o}}{N!} + (-)^N \delta_{\ell N}, \tag{D.7}$$

Expand the factor $(q^2)^{m\varepsilon}$ in (D.4) to $O(\varepsilon^N)$ and again use (D.7) to obtain the result

$$\int^{N\text{-loop}} 1 \sim \frac{1}{N!} \left[\frac{(-)^{N-1}}{\varepsilon^N} + \ell n^N q^2\right] + O(\varepsilon). \tag{D.8}$$

This result is not identical to that of the exponent derivative of order N, (cf. (4.16c)),

$$\frac{1}{(N-1)!} \lim_{\rho\to o} \left(\frac{d}{d\rho}\right)^{N-1} \frac{(q^2)^{\varepsilon+\rho}}{\varepsilon+\rho} = \frac{(-)^{N-1}}{\varepsilon^N} + \frac{1}{N!} \ell n^N q^2 + O(\varepsilon), \tag{D.9}$$

reflecting the fact that the 'tHooft-Veltman prescription corresponds to a special limiting order - first limit $\rho\to\varepsilon$, then limit $\varepsilon\to 0$ - of the more general hybrid method.

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